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JORDAN *k*-DERIVATIONS OF CERTAIN NOBUSAWA GAMMA RINGS

Sujoy Chakraborty[1]

Department of Mathematics
Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh
E-mail: sujoy_chbty@yahoo.com
and
Akhil Chandra Paul
Department of Mathematic
Rajshahi University, Rajshahi-6205, Bangladesh
E-mail: acpaulru_math@yahoo.com

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ABSTRACT

From the very definition, it follows that every Jordan *k*-derivation of a gamma ring *M* is, in general, not a *k*-derivation of *M*. In this article, we establish its generalization by considering *M* as a 2-torsion free semiprime Γ_N-ring (Nobusawa gamma ring). We also show that every Jordan *k*-derivation of a 2-torsion free completely semiprime Γ_N-ring is a *k*-derivation of the same.

1. Introduction

For the sake of completeness of the study, we begin with the following introductory definitions and examples.

Definition 1.1 *Let M and* Γ *be additive abelian groups. If there exists a mapping* $(a,\alpha,b)\to a\alpha b$ *of* $M\times\Gamma\times M\to M$ *such that the conditions*

(a) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,

and (b) $(a\alpha b)\beta c = a\alpha(b\beta c)$

are satisfied for all $a,b,c\in M$ *and* $\alpha,\beta\in\Gamma$ *, then M is said to be a gamma ring in the sense of* Barnes[1], *or simply, a gamma ring (symbolically,* Γ*-ring).*

Example 1.1 *If R is an ordinary associative ring, U is any ideal of R, and I is the ring of integers, then R is a* Γ*-ring with* $\Gamma = R$ *or,* $\Gamma = U$ *or,* $\Gamma = I$*. Also, U is a* Γ*-ring with* $\Gamma = R$*.*

Definition 1.2 *In addition to all the assumptions and conditions in the definition of a* Γ*-ring given above, if there is another mapping* $(\alpha,a,\beta)\to\alpha a\beta$ *of* $\Gamma\times M\times\Gamma\to\Gamma$ *such that the properties*

(a^*) $(\alpha+\beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a+b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta+\gamma) = \alpha a\beta + \alpha a\gamma$,

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[1]Corresponding author

(b^*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ *, and*

(c^*) $a\alpha b = 0$ *implies* $\alpha = 0$

hold for all $a, b, c \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$ *, then M is called a gamma ring in the sense of* Nobusawa[9], *or simply, a Nobusawa* Γ*-ring (symbolically,* Γ_N *-ring).*

Example 1.2 *If R is an ordinary associative ring with the unity 1, then R is a* Γ_N *-ring with* Γ*=R.*

The notions of derivation and Jordan derivation of a Γ-ring have been introduced by M. Sapanci and A. Nakajima [10] as follows.

Definition 1.3 *Let M be a* Γ*-ring, and let* $d : M \to M$ *be an additive mapping such that*

$$d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$$

is satisfied for all $a, b \in M$ *and* $\alpha \in \Gamma$ *; then d is called a derivation of M.*

Definition 1.4 *For a* Γ*-ring M, if* $d : M \to M$ *is an additive mapping such that*

$$d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$$

holds for all $a \in M$ *and* $\alpha \in \Gamma$ *, then d is said to be a Jordan derivation of M.*

In accordance with the notion of derivation of a Γ-ring mentioned as above, H. Kandamar [8] has introduced the concept of *k*-derivation of a Γ_N -ring as follows.

Definition 1.5 *Let M be a* Γ_N *-ring, and let* $d : M \to M$ *and* $k : \Gamma \to \Gamma$ *be additive mappings. If*

$$d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$$

is satisfied for all $a, b \in M$ *and* $\alpha \in \Gamma$ *, then d is called a k-derivation of M.*

Example 1.3 *Let M be a* Γ_N *-ring, and let* $a \in M$ *and* $\alpha \in \Gamma$ *be any two fixed elements. Define the additive mappings* $d : M \to M$ *and* $k : \Gamma \to \Gamma$ *by* $d(x) = a\alpha x$ *(for all* $x \in M$ *) and* $k(\beta) = \beta a\alpha$ *(for all* $\beta \in \Gamma$ *), respectively. Then d is a k-derivation of M, for*

$$d(x\beta y) = a\alpha(x\beta y) = a\alpha x\beta y - x\beta a\alpha y + x\beta a\alpha y$$

$$= (a\alpha x)\beta y - x(\beta a\alpha)y + x\beta(a\alpha y) = d(x)\beta y + xk(\beta)y + x\beta d(y) .$$

Now we introduce the concept of Jordan *k*-derivation of a Γ_N -ring using the notion of *k*-derivation of a Γ-ring due to H. Kandamar [8] as bellow.

Definition 1.6 *Let M be a* Γ_N *-ring, and let* $d : M \to M$ *and* $k : \Gamma \to \Gamma$ *be additive mappings. Then d is said to be a Jordan k-derivation of M if*

$$d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$$

holds for all $a \in M$ *and* $\alpha \in \Gamma$.

Example 1.4 *Let M be a* Γ_N*-ring, and let d be a k-derivation of M. Consider* $M_1 = \{(x,x) : x \in M\}$ *and* $\Gamma_1 = \{(\alpha,\alpha) : \alpha \in \Gamma\}$. *Let the operations of addition and multiplication on* M_1 *and* Γ_1 *be defined by*

$$(x_1,x_1)+(x_2,x_2)=(x_1+x_2,x_1+x_2),\ (x_1,x_1)(\alpha,\alpha)(x_2,x_2)=(x_1\alpha x_2,x_1\alpha x_2) \text{ and}$$

$$(\alpha_1,\alpha_1)+(\alpha_2,\alpha_2)=(\alpha_1+\alpha_2,\alpha_1+\alpha_2),\ (\alpha_1,\alpha_1)(x,x)(\alpha_2,\alpha_2)=(\alpha_1 x\alpha_2,\alpha_1 x\alpha_2)$$

for every $x,x_1,x_2 \in M$ *and* $\alpha,\alpha_1,\alpha_2 \in \Gamma$, *respectively. Then* M_1 *is clearly a Nobusawa* Γ_1*-ring under these operations. Let* $d_1 : M_1 \to M_1$ *and* $k_1 : \Gamma_1 \to \Gamma_1$ *be the additive mappings defined by*

$$d_1(x,x)=(d(x),d(x)) \text{ and } k_1(\alpha,\alpha)=(k(\alpha),k(\alpha))$$

for all $x \in M$ *and* $\alpha \in \Gamma$, *respectively. If we say* $(x,x)=a \in M$ *and* $(\alpha,\alpha)=\gamma \in \Gamma$ *for any* $x \in M$ *and* $\alpha \in \Gamma$, *then we have*

$$\begin{aligned} d_1(a\gamma a) &= d_1((x,x)(\alpha,\alpha)(x,x)) = (d(x\alpha x), d(x\alpha x)) \\ &= (d(x)\alpha x, d(x)\alpha x)+(xk(\alpha)x, xk(\alpha)x)+(x\alpha d(x), x\alpha d(x)) \\ &= d_1(a)\gamma a + ak_1(\gamma)a + a\gamma d_1(a) \end{aligned}$$

Hence, it follows that d_1 *is a Jordan* k_1*-derivation of M. Obviously,* d_1 *is not a* k_1*-derivation of M.*

Considering *M* as a Γ-ring (**until any further notice is mentioned hereafter in this section**), we recall some important definitions needful for us as follows:

Definition 1.7 *An additive subgroup U of M is called a left (resp., right) ideal of M if* $M\Gamma U \subset U$ *(resp.,,* $U\Gamma M \subset U$*). U is called a two-sided ideal, or simply, an ideal of M if U is a left as well as a right ideal of M (that is, if both* $m\gamma u \in U$ *and* $u\gamma m \in U$ *for all* $m \in M$, $\gamma \in \Gamma$ *and* $u \in U$*).*

Definition 1.8 *M is said to be a 2-torsion free* Γ*-ring if* $2a = 0$ *implies* $a = 0$ *for all* $a \in M$. *Besides, M is called a commutative* Γ*-ring if* $x\gamma y = y\gamma x$ *holds for all* $x, y \in M$ *and* $\gamma \in \Gamma$. *The set* $Z(M) = \{c \in M : c\alpha m = m\alpha c \text{ for all } \alpha \in \Gamma \text{ and } m \in M\}$ *is known as the center of the* Γ*-ring M.*

Definition 1.9 *If* $a,b \in M$ *and* $\alpha \in \Gamma$, *then* $[a,b]_\alpha = a\alpha b - b\alpha a$ *is called the commutator of a and b with respect to* α.

Lemma 1.1 *If M is a* Γ*-ring, then, for all* $a,b,c \in M$ *and* $\alpha,\beta \in \Gamma$:

(i) $[a,b]_\alpha + [b,a]_\alpha = 0$; (ii) $[a+b,c]_\alpha = [a,c]_\alpha + [b,c]_\alpha$;

(iii) $[a,b+c]_\alpha = [a,b]_\alpha + [a,c]_\alpha$; (iv) $[a,b]_{\alpha+\beta} = [a,b]_\alpha + [a,b]_\beta$.

Remark 1.1 *A necessary and sufficient condition for a* Γ*-ring M to be commutative is that* $[a,b]_\alpha = 0$ *for all* $a,b \in M$ *and* $\alpha \in \Gamma$.

Definition 1.10 *An element* $x \in M$ *is called a nilpotent element if (for any* $\gamma \in \Gamma$*), there exists a positive integer n (depending on* γ*) such that* $(x\gamma)^n x = (x\gamma)(x\gamma)...(x\gamma)x = 0$. *Besides, an ideal U of M is said to be a nil ideal if each element of U is nilpotent. Moreover, an ideal I of M is called a nilpotent ideal if there exists a positive integer n such that* $(I\Gamma)^n I = (I\Gamma)(I\Gamma)...(I\Gamma)I = 0$.

Remark 1.2 *Every nilpotent ideal of a* Γ*-ring is nil.*

Definition 1.11 (i) *M is called prime if* $a\Gamma M\Gamma b = 0$ *(with* $a,b \in M$*) implies a*=0 *or b*=0*;* (ii) *M is said to be completely prime if* $a\Gamma b = 0$ *(with* $a,b \in M$*) implies a*=0 *or b*=0*;* (iii) *M is called semiprime if* $a\Gamma M\Gamma a = 0$ *(with* $a \in M$*) implies a*=0*;* (iv) *M is said to be completely semiprime if* $a\Gamma a = 0$ *(with* $a \in M$*) implies a*=0*.*

Remark 1.3 *Every prime* Γ*-ring is semiprime, and also, every completely prime* Γ*-ring is completely semiprime.*

2. Some Consequences

We now state some useful results without their proofs, because all of these results (in this section) have already been proved in our papers [4] and [5].

Lemma 2.1 *Let M be a* Γ_N*-ring and let d be a Jordan k-derivation of M. Then for all* $a,b,c \in M$ *and* $\alpha,\beta \in \Gamma$*, the following statements hold:*

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha d(b) + b\alpha d(a)$;

(ii) $$d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha bk(\beta)a + a\beta bk(\alpha)a + a\alpha b\beta d(a) + a\beta b\alpha d(a).$$

In particular, if M is 2-torsion free, then

(iii) $d(a\alpha b\alpha a) = d(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha d(b)\alpha a + a\alpha bk(\alpha)a + a\alpha b\alpha d(a)$;

(iv) $$d(a\alpha b\alpha c + c\alpha b\alpha a) = d(a)\alpha b\alpha c + d(c)\alpha b\alpha a + ak(\alpha)b\alpha c + ck(\alpha)b\alpha a + a\alpha d(b)\alpha c + c\alpha d(b)\alpha a + a\alpha bk(\alpha)c + c\alpha bk(\alpha)a + a\alpha b\alpha d(c) + c\alpha b\alpha d(a).$$

Lemma 2.2 *Let d be a Jordan k-derivation of a 2-torsion free* Γ_N*-ring M. Then, for all* $b \in M$ *and* $\beta \in \Gamma$, $k(\beta b\beta) = k(\beta)b\beta + \beta d(b)\beta + \beta bk(\beta)$.

Lemma 2.3 *If d is a Jordan* k_1*-derivation as well as a Jordan* k_2*-derivation of a 2-torsion free* Γ_N*-ring M, then* $k_1 = k_2$.

Remark 2.1 *k is uniquely determined if d is a Jordan k-derivation of a 2-torsion free* Γ_N*-ring.*

Definition 2.1 *Let d be a Jordan k-derivation of a* Γ_N *-ring M. If* $a,b \in M$ *and* $\alpha \in \Gamma$ *, then we define* $F_\alpha(a,b) = d(a\alpha b) - d(a)\alpha b - ak(\alpha)b - a\alpha d(b)$.

Lemma 2.4 *If d is a Jordan k-derivation of a* Γ_N *-ring M, then for all* $a,b,c \in M$ *and* $\alpha,\beta \in \Gamma$,

(i) $F_\alpha(a,b) + F_\alpha(b,a) = 0$; (ii) $F_\alpha(a+b,c) = F_\alpha(a,c) + F_\alpha(b,c)$;

(iii) $F_\alpha(a,b+c) = F_\alpha(a,b) + F_\alpha(a,c)$; (iv) $F_{\alpha+\beta}(a,b) = F_\alpha(a,b) + F_\beta(a,b)$.

Remark 2.2 *d is a k-derivation of a* Γ_N *-ring M if and only if* $F_\alpha(a,b) = 0$ *for all* $a,b \in M$ *and* $\alpha \in \Gamma$.

3. Jordan *k*-Derivations of Semiprime Γ_N -Rings

In classical ring theory, I. N. Herstein [7] has shown that every Jordan derivation of a 2-torsion free prime ring is a derivation of the same. The similar result for 2-torsion free semiprime rings has been proved by M. Bresar [2]. Here we extend this result for a 2-torsion free semiprime Γ_N -ring to show that every Jordan *k*-derivation of a 2-torsion free semiprime Γ_N -ring *M* is a *k*-derivation of *M*.

Lemma 3.1 *Let M be a semiprime* Γ*-ring. Then M contains no nonzero nilpotent ideal.*

Proof. Let *I* be a nilpotent ideal of *M*. Then $(I\Gamma)^n I = 0$ for some positive integer *n*. Let us assume that *n* is minimum. Now suppose that $n > 1$. Since $I\Gamma M \subset I$, we then have

$$(I\Gamma)^{n-1} I\Gamma M\Gamma (I\Gamma)^{n-1} I \subset (I\Gamma)^{n-1} I\Gamma (I\Gamma)^{n-1} I = (I\Gamma)^n I\Gamma (I\Gamma)^{n-2} I = 0 .$$

Hence, by the semiprimeness of *M*, we get $(I\Gamma)^{n-1} I = 0$, a contradiction to the minimality of *n*. Therefore, $n = 1$. Thus, $I\Gamma I = 0$. Then $I\Gamma M\Gamma I \subset I\Gamma I = 0$. Since *M* is semiprime, it gives $I = 0$.

But, since every prime Γ-ring is semiprime, we have:

Corollary 3.1 *Every prime* Γ*-ring has no nonzero nilpotent ideal.*

Again, since every nilpotent ideal of a Γ-ring is nil, it follows that

Corollary 3.2 *Semiprime (and also, prime)* Γ*-rings have no nonzero nilpotent element.*

Lemma 3.2 *The center of a semiprime (or, prime)* Γ*-ring does not contain any nonzero nilpotent element.*

Proof. Let *Z* be the center of a semiprime Γ-ring *M*. Then *Z* is a subring of *M* (as we know). Thus, since *M* is a semiprime Γ-ring, *Z* is so. Hence, by Corollary 3.2, *Z* has no nonzero nilpotent element. A similar reason proves the claim for a prime Γ-ring.

Lemma 3.3 *Let d be a Jordan k-derivation of a 2-torsion free* Γ_N *-ring M. Then*

(i) $F_\alpha(a,b)\alpha m\alpha[a,b]_\alpha + [a,b]_\alpha \alpha m\alpha F_\alpha(a,b) = 0$ *and*

(ii) $F_\alpha(a,b)\beta m\beta[a,b]_\alpha + [a,b]_\alpha \beta m\beta F_\alpha(a,b) = 0$

for all $a,b \in M$ *and* $\alpha \in \Gamma$.

Proof. We have proved this lemma in our paper [4].

Lemma 3.4 *Let M be a 2-torsion free semiprime* Γ*-ring. If* $a,b,m \in M$ *and* $\alpha,\beta \in \Gamma$ *such that* $a\alpha m\beta b + b\alpha m\beta a = 0$*, then* $a\alpha m\beta b = b\alpha m\beta a = 0$.

Proof. Let $x \in M$ and $\gamma,\delta \in \Gamma$ be arbitrary elements. By using $a\alpha m\beta b = -b\alpha m\beta a$ (where $a,b,m \in M$ and $\alpha,\beta \in \Gamma$) repeatedly, we get

$$(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b) = -(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b).$$

This implies, $2((a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b)) = 0$.

Since *M* is 2-torsion free, it gives $(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b) = 0$;

that is, we have $(a\alpha m\beta b)\Gamma M\Gamma(a\alpha m\beta b) = 0$.

But, since *M* is semiprime, we obtain $a\alpha m\beta b = 0$. Hence, $a\alpha m\beta b = b\alpha m\beta a = 0$.

Corollary 3.3 *Let M be a 2-torsion free semiprime* Γ_N *-ring. Then, for all* $a,b,m \in M$ *and* $\alpha,\beta \in \Gamma$,

(i) $F_\alpha(a,b)\alpha m\alpha[a,b]_\alpha = 0$; (ii) $[a,b]_\alpha \alpha m\alpha F_\alpha(a,b) = 0$;

(iii) $F_\alpha(a,b)\beta m\beta[a,b]_\alpha = 0$; (iv) $[a,b]_\alpha \beta m\beta F_\alpha(a,b) = 0$.

Proof. Applying Lemma 3.4 in Lemma 3.3, we obtain the required results.

Lemma 3.5 *Let M be a 2-torsion free semiprime* Γ_N *-ring. Then, for all* $a,b,u,v,m \in M$ *and* $\alpha,\beta \in \Gamma$,

(i) $F_\alpha(a,b)\beta m\beta[u,v]_\alpha = 0$; (ii) $[u,v]_\alpha \beta m\beta F_\alpha(a,b) = 0$;

(iii) $F_\alpha(a,b)\beta m\beta[u,v]_\gamma = 0$; (iv) $[u,v]_\gamma \beta m\beta F_\alpha(a,b) = 0$.

Proof. (i) Replacing $a + u$ for a in Corollary 3.3(iii), we obtain

$$F_\alpha(a,b)\beta m\beta[u,b]_\alpha = -F_\alpha(a,b)\beta m\beta[a,b]_\alpha.$$

Therefore, we have

$$(F_\alpha(a,b)\beta m\beta[u,b]_\alpha)\beta m\beta(F_\alpha(a,b)\beta m\beta[u,b]_\alpha)$$
$$= -F_\alpha(a,b)\beta m\beta[u,b]_\alpha \beta m\beta F_\alpha(u,b)\beta m\beta[a,b]_\alpha = 0.$$

Hence, by the semiprimeness of *M*, we get $F_\alpha(a,b)\beta m\beta[u,b]_\alpha = 0$. Similarly, by replacing $b + v$ for b in this equality, we obtain $F_\alpha(a,b)\beta m\beta[u,v]_\alpha = 0$.

(ii) Proceeding in the same way as above by the similar replacements successively in Corollary 3.3(iv), we obtain $[u,v]_\alpha \beta m \beta F_\alpha(a,b) = 0$ for all $a,b,u,v,m \in M$ and $\alpha,\beta \in \Gamma$.

(iii) Putting $\alpha+\gamma$ for α in (i), we get $F_\alpha(a,b)\beta m \beta [u,v]_\gamma = -F_\gamma(a,b)\beta m \beta [u,v]_\alpha$. Then

$$(F_\alpha(a,b)\beta m \beta [u,v]_\gamma)\beta m \beta (F_\alpha(a,b)\beta m \beta [u,v]_\gamma)$$
$$= -F_\alpha(a,b)\beta m \beta [u,v]_\gamma \beta m \beta F_\gamma(a,b)\beta m \beta [u,v]_\alpha = 0.$$

By the semiprimeness of *M*, we have $F_\alpha(a,b)\beta m \beta [u,v]_\gamma = 0$.

(iv) As in the proof of (iii), the similar replacement in (ii) produces (iv).

Theorem 3.1 *Every Jordan k-derivation of a 2-torsion free semiprime Γ_N-ring M is a k-derivation of M.*

Proof. Let *d* be a Jordan *k*-derivation of a 2- torsion free semiprime Γ_N-ring *M*. Let $a,b,u,v,m \in M$ and $\alpha,\beta \in \Gamma$. Then, by Lemma 3.5(iii), we obtain

$$[F_\alpha(a,b),v]_\beta \beta m \beta [F_\alpha(a,b),v]_\beta$$
$$= (F_\alpha(a,b)\beta v - v\beta F_\alpha(a,b))\beta m \beta [F_\alpha(a,b),v]_\beta$$
$$= F_\alpha(a,b)\beta v \beta m \beta [F_\alpha(a,b),v]_\beta - v\beta F_\alpha(a,b)\beta m \beta [F_\alpha(a,b),v]_\beta = 0,$$

since $v\beta m \in M$ and $F_\alpha(a,b) \in M$ for all $a,b,v,m \in M$ and $\alpha,\beta \in \Gamma$.

Therefore, we get $[F_\alpha(a,b),v]_\beta = 0$ (by the semiprimeness of *M*). But since $F_\alpha(a,b) \in M$ (for all $a,b \in M$ and $\alpha \in \Gamma$), it follows that $F_\alpha(a,b) \in Z(M)$.

Now let $\gamma,\delta \in \Gamma$. By Lemma 3.5(ii), $F_\alpha(a,b)\gamma[u,v]_\alpha \delta m \delta F_\alpha(a,b)\gamma[u,v]_\alpha = 0$. But, since *M* is semiprime, we get

$$F_\alpha(a,b)\gamma[u,v]_\alpha = 0. \tag{1}$$

Also, by Lemma 3.5(i), we have $[u,v]_\alpha \gamma F_\alpha(a,b)\delta m \delta [u,v]_\alpha \gamma F_\alpha(a,b) = 0$, and hence, the semiprimeness of *M* implies that

$$[u,v]_\alpha \gamma F_\alpha(a,b) = 0. \tag{2}$$

Similarly, by Lemma 3.5(iv), we get $F_\alpha(a,b)\gamma[u,v]_\beta \delta m \delta F_\alpha(a,b)\gamma[u,v]_\beta = 0$. Since *M* is semiprime, it follows that

$$F_\alpha(a,b)\gamma[u,v]_\beta = 0. \tag{3}$$

Again, by Lemma 3.5(iii), we have $[u,v]_\beta \gamma F_\alpha(a,b)\delta m \delta [u,v]_\beta \gamma F_\alpha(a,b) = 0$, and therefore, by the semiprimeness of *M*, we obtain

$$[u,v]_\beta \gamma F_\alpha(a,b) = 0. \tag{4}$$

Thus, we have

$$2F_\alpha(a,b)\gamma F_\alpha(a,b)=F_\alpha(a,b)\gamma(F_\alpha(a,b)+F_\alpha(a,b))=F_\alpha(a,b)\gamma(F_\alpha(a,b)-F_\alpha(a,b))$$
$$=F_\alpha(a,b)\gamma d([a,b]_\alpha)-F_\alpha(a,b)\gamma[d(a),b]_\alpha)-F_\alpha(a,b)\gamma[a,d(b)]_\alpha)-F_\alpha(a,b)\gamma[a,b]_{k(\alpha)}.$$

Since $d(a),d(b)\in M$ and $k(\alpha)\in\Gamma$, by using (1) and (3), we get

$$F_\alpha(a,b)\gamma[d(a),b]_\alpha)=F_\alpha(a,b)\gamma[a,d(b)]_\alpha)=F_\alpha(a,b)\gamma[a,b]_{k(\alpha)}=0,$$

and therefore,

$$2F_\alpha(a,b)\gamma F_\alpha(a,b)=F_\alpha(a,b)\gamma d([a,b]_\alpha). \tag{5}$$

By the operation (3) + (4), we obtain

$$F_\alpha(a,b)\gamma[u,v]_\beta+[u,v]_\beta\gamma F_\alpha(a,b)=0.$$

Then Lemma 2.1(i), equation (3) and $F_\alpha(a,b)\in Z(M)$ gives

$$0=d(F_\alpha(a,b)\gamma[u,v]_\beta+[u,v]_\beta\gamma F_\alpha(a,b))$$
$$=d(F_\alpha(a,b))\gamma[u,v]_\beta+d([u,v]_\beta)\gamma F_\alpha(a,b)+F_\alpha(a,b)k(\gamma)[u,v]_\beta$$
$$+[u,v]_\beta k(\gamma)F_\alpha(a,b)+F_\alpha(a,b)\gamma d([u,v]_\beta)+[u,v]_\beta\gamma d(F_\alpha(a,b))$$
$$=d(F_\alpha(a,b))\gamma[u,v]_\beta+2F_\alpha(a,b)\gamma d([u,v]_\beta)+[u,v]_\beta\gamma d(F_\alpha(a,b)).$$

Therefore, we get

$$2F_\alpha(a,b)\gamma d([u,v]_\beta)=-d(F_\alpha(a,b))\gamma[u,v]_\beta-[u,v]_\beta\gamma d(F_\alpha(a,b)). \tag{6}$$

From (5) and (6), we then have

$$4F_\alpha(a,b)\gamma F_\alpha(a,b)=2F_\alpha(a,b)\gamma d([a,b]_\alpha)$$
$$=-d(F_\alpha(a,b))\gamma[a,b]_\beta-[a,b]_\beta\gamma d(F_\alpha(a,b)).$$

Thus, we obtain

$$4F_\alpha(a,b)\gamma F_\alpha(a,b)\gamma F_\alpha(a,b)$$
$$=-d(F_\alpha(a,b))\gamma[a,b]_\beta\gamma F_\alpha(a,b)-[a,b]_\beta\gamma d(F_\alpha(a,b))\gamma F_\alpha(a,b).$$

Here, it follows that $d(F_\alpha(a,b))\gamma[a,b]_\beta\gamma F_\alpha(a,b)=0$, since $[a,b]_\beta\gamma F_\alpha(a,b)=0$ (by (4)); and also, $[a,b]_\beta\gamma d(F_\alpha(a,b))\gamma F_\alpha(a,b)=0$ (by Lemma 3.5(iv)), since $d(F_\alpha(a,b))\in M$ for all $a,b\in M$ and $\alpha\in\Gamma$. Therefore, we obtain $4F_\alpha(a,b)\gamma F_\alpha(a,b)\gamma F_\alpha(a,b)=0$. That is, we have $4(F_\alpha(a,b)\gamma)^2F_\alpha(a,b)=0$. So, $(F_\alpha(a,b)\gamma)^2F_\alpha(a,b)=0$ (since *M* is 2-torsion free). Thus, $F_\alpha(a,b)$ is a nilpotent element of the Γ_N -ring *M*. But, we know that the center of a semiprime Γ_N-ring does not contain any nonzero nilpotent element (by Lemma 3.2). Hence, $F_\alpha(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$. It means, *d* is then a *k*-derivation of *M*.

4. Jordan *k*-Derivations of Completely Semiprime Γ_N-Rings

In sequel to the last result, we now prove it analogously in case of a 2-torsion free completely semiprime Γ_N-ring. To reach our goal in this section, we develop some useful results in the following way.

Lemma 4.1 *A completely semiprime* Γ*-ring has no nonzero nilpotent ideal.*

Proof. Let *I* be an ideal of *M* such that $(I\Gamma)^n I=0$ for some positive integer *n*. Assume that *n* is minimum and that $n>1$. Then $(I\Gamma)^{n-1}I\Gamma(I\Gamma)^{n-1}I=(I\Gamma)^n I\Gamma(I\Gamma)^{n-2}I=0$. Since *M* is completely semiprime, we get $(I\Gamma)^{n-1}I=0$, which is a contradiction to the minimality of *n*. Hence, we conclude that $n=1$. Thus, we obtain $I\Gamma I=0$. So, the completely semiprimeness of *M* implies that $I=0$.

But, since every completely prime Γ-ring is completely semiprime, we have:

Corollary 4.1 *A completely prime* Γ*-ring has no nonzero nilpotent ideal.*

Again, since every nilpotent ideal of a Γ-ring is nil, it follows that

Corollary 4.2 *Completely semiprime (and also, completely prime)* Γ*-rings have no nonzero nilpotent element.*

Lemma 4.2 *The center of a completely semiprime (or, completely prime)* Γ*-ring does not contain any nonzero nilpotent element.*

Proof. If *Z* is the center of a completely semiprime Γ-ring *M*, then we know that *Z* is a subring of *M*. Since *M* is completely semiprime, *Z* is also a completely semiprime Γ-ring. So, by Corollary 4.2, *Z* has no nonzero nilpotent element. It also proves the claim for a completely prime Γ-ring similarly.

Lemma 4.3 *Let d be a Jordan k-derivation of a* Γ_N*-ring M, and suppose that* $a,b\in M$ *and* $\alpha,\gamma\in\Gamma$. *Then* $F_\alpha(a,b)\gamma[a,b]_\alpha+[a,b]_\alpha\gamma F_\alpha(a,b)=0$.

Proof. This result is proved in our paper [5].

Lemma 4.4 *Let M be a 2-torsion free completely semiprime* Γ_N*-ring, and suppose* $a,b\in M$ *and* $\gamma\in\Gamma$ *such that* $a\gamma b+b\gamma a=0$. *Then* $a\gamma b=b\gamma a=0$.

Proof. Suppose that δ is an arbitrary element of Γ. Let $a,b\in M$ and $\gamma\in\Gamma$ such that $a\gamma b+b\gamma a=0$. Hence, by using $a\gamma b=-b\gamma a$ repeatedly, we get

$$(a\gamma b)\delta(a\gamma b)=-(b\gamma a)\delta(a\gamma b)=-(b(\gamma a\delta)a)\gamma b=(a(\gamma a\delta)b)\gamma b$$
$$=a\gamma(a\delta b)\gamma b=-a\gamma(b\delta a)\gamma b=-(a\gamma b)\delta(a\gamma b)$$

This implies, $2((a\gamma b)\delta(a\gamma b))=0$. Since M is 2-torsion free, we have $(a\gamma b)\delta(a\gamma b)=0$; that is, $(a\gamma b)\Gamma(a\gamma b)=0$. By the completely semiprimeness of M, we get $a\gamma b=0$. Hence, $a\gamma b=b\gamma a=0$.

Corollary 4.3 *Let M be a 2-torsion free completely semiprime* Γ_N *-ring. Then, for all* $a,b\in M$ *and* $\alpha,\gamma\in\Gamma$,

(i) $F_\alpha(a,b)\gamma[a,b]_\alpha=0$; (ii) $[a,b]_\alpha\gamma F_\alpha(a,b)=0$.

Proof. By applying Lemma 4.4 in the result of Lemma 4.3, we obtain this corollary.

Lemma 4.5 *Let M be a 2-torsion free completely semiprime* Γ_N *-ring. Then, for all* $a,b,u,v,m\in M$ *and* $\alpha,\gamma\in\Gamma$,

(i) $F_\alpha(a,b)\gamma[u,v]_\alpha=0$; (ii) $[u,v]_\alpha\gamma F_\alpha(a,b)=0$;

(iii) $F_\alpha(a,b)\gamma[u,v]_\beta=0$; (iv) $[u,v]_\beta\gamma F_\alpha(a,b)=0$.

Proof. (i) Replacing *a* + *u* for *a* in Corollary 4.3(i), we get

$$F_\alpha(a,b)\gamma[u,b]_\alpha=-F_\alpha(u,b)\gamma[a,b]_\alpha.$$

Hence, we have $F_\alpha(a,b)\gamma[u,b]_\alpha\gamma F_\alpha(a,b)\gamma[u,b]_\alpha=0$. By the completely semiprimeness of M, we obtain $F_\alpha(a,b)\gamma[u,b]_\alpha=0$. Similarly, by replacing *b* + *v* for *b* in this equality obtained, we get $F_\alpha(a,b)\gamma[u,v]_\alpha=0$.

(ii) The similar replacements (as above) in Corollary 4.3(ii) yields $[u,v]_\alpha\gamma F_\alpha(a,b)=0$.

(iii) Putting $\alpha+\beta$ for α in (i), we get $F_\alpha(a,b)\gamma[u,v]_\beta=-F_\beta(a,b)\gamma[u,v]_\alpha$ which then implies that $F_\alpha(a,b)\gamma[u,v]_\beta\gamma F_\alpha(a,b)\gamma[u,v]_\beta=0$. But, since M is completely semiprime, we obtain $F_\alpha(a,b)\gamma[u,v]_\beta=0$.

(iv) By performing the similar replacement in (ii) [as in the proof of (iii)], we easily get this required result.

Theorem 4.1 *If d is a Jordan k-derivation of a 2-torsion free completely semiprime* Γ_N *-ring M, then d is also a k-derivation of M.*

Proof. Let *d* be a Jordan *k*-derivation of a 2-torsion free completely semiprime Γ_N -ring *M*, and suppose that $a,b,v\in M$ and $\alpha,\beta,\gamma\in\Gamma$. Then, by using Lemma 4.5(iii), we have

$$[F_\alpha(a,b),v]_\beta\gamma[F_\alpha(a,b),v]_\beta=(F_\alpha(a,b)\beta v-v\beta F_\alpha(a,b))\gamma[F_\alpha(a,b),v]_\beta$$
$$=F_\alpha(a,b)\beta v\gamma[F_\alpha(a,b),v]_\beta-v\beta F_\alpha(a,b)\gamma[F_\alpha(a,b),v]_\beta=0,$$

since $\beta v\gamma\in M$ and $F_\alpha(a,b)\in M$ for all $a,b,v\in M$ and $\alpha,\beta,\gamma\in\Gamma$. So, $[F_\alpha(a,b),v]_\beta=0$ (since M is completely semiprime), where $F_\alpha(a,b)\in M$ for all $a,b,v\in M$ and $\alpha,\beta\in\Gamma$.

Therefore, we get $F_\alpha(a,b) \in Z(M)$.

Now, from Lemma 4.5(iii), we have

$$F_\alpha(a,b)\gamma[u,v]_\beta = 0\,. \tag{7}$$

And, from Lemma 4.5(iv), we get

$$[u,v]_\beta \gamma F_\alpha(a,b) = 0\,. \tag{8}$$

Thus, we obtain

$$2F_\alpha(a,b)\gamma F_\alpha(a,b) = F_\alpha(a,b)\gamma(F_\alpha(a,b) - F_\alpha(a,b))$$

$$= F_\alpha(a,b)\gamma d([a,b]_\alpha) - F_\alpha(a,b)\gamma[d(a),b]_\alpha) - F_\alpha(a,b)\gamma[a,d(b)]_\alpha) - F_\alpha(a,b)\gamma[a,b]_{k(\alpha)}\,.$$

Since $d(a), d(b) \in M$ and $k(\alpha) \in \Gamma$, by using Lemma 4.5(i) and (7), we get

$$F_\alpha(a,b)\gamma[d(a),b]_\alpha) = F_\alpha(a,b)\gamma[a,d(b)]_\alpha) = F_\alpha(a,b)\gamma[a,b]_{k(\alpha)} = 0\,,$$

and so, we have

$$2F_\alpha(a,b)\gamma F_\alpha(a,b) = F_\alpha(a,b)\gamma d([a,b]_\alpha)\,. \tag{9}$$

By the operation (7) + (8), we get $F_\alpha(a,b)\gamma[u,v]_\beta + [u,v]_\beta \gamma F_\alpha(a,b) = 0$. Then, by Lemma 2.1(i) with the use of (7), and since $F_\alpha(a,b) \in Z(M)$, we have

$$0 = d(F_\alpha(a,b)\gamma[u,v]_\beta + [u,v]_\beta \gamma F_\alpha(a,b))$$

$$= d(F_\alpha(a,b))\gamma[u,v]_\beta + 2F_\alpha(a,b)\gamma d([u,v]_\beta) + [u,v]_\beta \gamma d(F_\alpha(a,b))\,.$$

Therefore, we get

$$2F_\alpha(a,b)\gamma d([u,v]_\beta) = -d(F_\alpha(a,b))\gamma[u,v]_\beta - [u,v]_\beta \gamma d(F_\alpha(a,b))\,. \tag{10}$$

Then, from (9) and (10), we have

$$4F_\alpha(a,b)\gamma F_\alpha(a,b) = 2F_\alpha(a,b)\gamma d([a,b]_\alpha)$$

$$= -d(F_\alpha(a,b))\gamma[a,b]_\beta - [a,b]_\beta \gamma d(F_\alpha(a,b))\,.$$

Thus, we obtain

$$4F_\alpha(a,b)\gamma F_\alpha(a,b)\gamma F_\alpha(a,b)$$

$$= -d(F_\alpha(a,b))\gamma[a,b]_\beta \gamma F_\alpha(a,b) - [a,b]_\beta \gamma d(F_\alpha(a,b))\gamma F_\alpha(a,b)\,.$$

Now, by (8), we get $d(F_\alpha(a,b))\gamma[a,b]_\beta \gamma F_\alpha(a,b) = 0$ (since $[a,b]_\beta \gamma F_\alpha(a,b) = 0$); and by Lemma 4.5(iv), we obtain $[a,b]_\beta \gamma d(F_\alpha(a,b))\gamma F_\alpha(a,b) = 0$ (since $d(F_\alpha(a,b)) \in M$). Thus, $4(F_\alpha(a,b)\gamma)^2 F_\alpha(a,b) = 0$. Since M is 2-torsion free, it gives $(F_\alpha(a,b)\gamma)^2 F_\alpha(a,b) = 0$.

So, $F_\alpha(a,b)$ is a nilpotent element of the Γ_N-ring *M*, where $F_\alpha(a,b) \in Z(M)$. Hence, by Lemma 4.2, we get $F_\alpha(a,b) = 0$ (for all $a,b \in M$ and $\alpha \in \Gamma$) from which we conclude that *d* is a *k*-derivation of *M*.

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