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THE ENDOMORPHISM SEMIGROUP OF AN ENDOMAPPING OF A FINITE SET

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ABSTRACT

The structure of the semigroup of all endomorphisms of an endomapping of a finite set has been determined. This has been done by naturally representing the endomapping by a directed graph, and determining the structure of the endomorphism semigroup of this graph.

Keywords: Endomorphism semigroup, Transformation semigroup, Full transformation semigroup, Direct product, Wreath product, Directed graph.

1. Introduction

Let *X* be a finite non-epmty set and let $f : X \to X$ be an endomapping of *X*. The set of all endomappings of *X* is a semigroup under the composition of maps and is called the full transformation semigroup on *X* and is denoted by *E*(*X*). If the number of elements of *X* is *n*, we shall also write F_n for $E(X)$. A map $g : X \to X$ is called an endomorphism of *f* if $g f = f g$ i.e., if g belongs to the centraliser of *f* in *E*(*X*). The centraliser of *f*, $C(f) = \{g \in E(X) \mid g f = f g\}$, is a transformation semigroup on *X* and is called the endomorphism semigroup of *f*. We denote this semigroup by *End f*. In this paper, we shall determine the structure of this semigroup *End f* for a class of endomappings *f* such that, for each $x \in X$, there exists a positive integer r_x with the property that $f^{r_x+1}(x) = f^{r_x}(x)$. The technique of structure-determination consists of

(i) representing *f* by a directed graph $G(f)$ with vertices the points of *X* and edges $x \to f(x)$, and

(ii) determining the structure of the semigroup $End(G(f))$ of those transformations T of this directed graph $G(f)$ such that $T(f(x)) = f(T(x))$ i.e., $T(x \to f(x)) = (T(x) \to T(f(x)))$.

Since *T* maps vertices onto vertices and edges onto corresponding edges, *T* is called an endomorphism of the digraph of *f*. If g is the endomapping of *X* induced by *T*, the map $g \to T$ is an isomorphism of $End\, f$ into the endomorphism semigroup of $G(f)$. Then $G(f)$ has an appearance of the type:

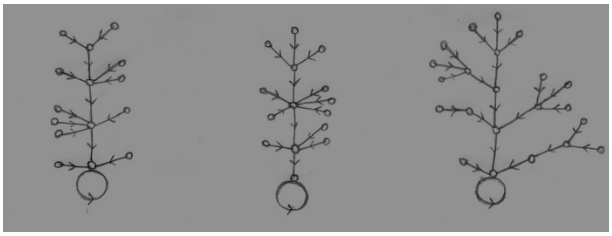

Fig.1

We shall determine the structure of the transformation semigroup *End*(*G*(*X*)) for a class of endomappings f. The discussion in the above ensures that the structure of *End f* will be known through the isomorphism $End\ f \cong End\,(G(X))$.

2. Necessary Preliminaries

To determine the structure of the transformation semigroup *End*(*G*(*X*)) we need some results of [4] about the direct product and wreath product of transformation semigroups. We recall these in the following:

A semigroup *S* is called a transformation semigroup on a nonempty set X and is written (S, X) if there is a map $S \times X \to X$ given by $(s, x) \to x$ such that $(s_1 s_2)(x) = s_1(s_2(x))$. If S is a monoid, then $1(x) = x$, for each $x \in X$. For transformation semigroups S_1 and S_2 on disjoint non-empty sets X_1, X_2, the direct product $S_1 \times S_2$ is a transformation semigroup on $X_1 \cup X_2$ with action given by $(s_1, s_2)(x_1) = s_1(x_1)$ and $(s_1, s_2)(x_2) = s_2(x_2)$.

For two non-empty sets X_1, X_2, the wreath product $S_1 \varsigma S_2$ is a transformation semigroup on $X_1 \times X_2$ and consists of maps $\theta\colon X_1, X_2 \to X_1, X_2$ given by $\theta(x_1, x_2) = (s_{1,x_2}(x_1), s_2(x_2))$, s_{1,x_2} being an element of S_1 determined by x_2.

The following results in [4] show that (i) wreath product has a description in terms of direct product which makes the sense that wreath product is associative and is distributive over direct product.

Theorem 2.1

$$(S_1 \varsigma S_2, X \times X_2) \cong ((\bigtimes_{x_2 \in X_2} S_{1,x_2}) \times S_2,\ (\bigcup_{x_2 \in X_2} X_{1,x_2}) \times X_2)$$

where each $x_2 \in X_2$, $S_{1,x_2} \cong S_1$ and $|X_{1,x_2}| = |X_1|$.

Theorem 2.2 $((S_1 \wr S_2) \wr S_3,\ (X_1 \times X_2) \times X_3) \cong (S_1 \wr (S_2 \wr S_3),\ X_1 \times (X_2 \times X_3))$.

Theorem 2.3

$(S_1 \wr (S_2 \times S_3),\ X_1 \times (X_2 \cup X_3)) \cong ((S_1 \wr S_2) \times (S_1 \wr S_3),\ (X_1 \times X_2) \cup (X_1 \times X_3))$.

Remarks.

(i) If $S_2 = \{1_{X_2}\}$, then $(S_1 \times S_2,\ X_1 \cup X_2)$ may be identified with (S_1, X_1) and $(S_1 \wr S_2,\ X_1 \times X_2)$ with $(\prod_{x_2 \in X_2} S_{1,\,x_2},\ \bigcup_{x_2 \in X_2} X_{1,\,x_2})$ where $|X_{1,\,x_2}| = |X_1|$.

(ii) If $S_1 = \{1_{X_1}\}$, then both $(S_1 \times S_2,\ X_1 \cup X_2)$ and $(S_1 \wr S_2,\ X_1 \times X_2)$ may be identified with $(S_2,\ X_2)$.

(iii) If $X_1 = X_2 = X$, then $(S_1 \wr S_2, X \times X)$ may be identified with $(\prod_{x \in X} S_{1,\,x}) \times S_2,\ \bigcup_{x \in X} X_{1,\,x} \cup X)$. As semigroups, $S_1 \wr S_2 \cong (\prod_{x \in X} S_{1,x}) \times S_2$.

3. Structure of the endomorphism semigroup End *f*

We now determine the structure of $End\ f$ through representation of *f* by a directed graph. We begin with the following lemma:

Lemma 3.1 Let $G(f)$, the directed graph of *f*, be given by:

Then $End\ f \cong E(m) = \{\sigma_0, \sigma_1, \sigma_1^2 \cdots\cdots, \sigma_1^m\}$, is a cyclic semigroup generated by σ_1 and adjoined with an identity element of σ_0, with σ_1^m as the zero element, i.e., $\sigma_1^m \sigma_1^i = \sigma_1^m, \quad 0 \le i \le m.$

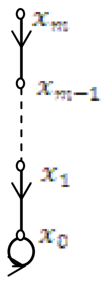

Fig.2

Proof. We observe that $g \in End\ f$ if and only if (i) $g(x_0) = x_0$ and if (ii) r is the positive integer such that $g = 1_X$ or $g(x_r) = x_0$, then $g(x_i) = x_0$, for each $i \le r$, and $g(x_s) = x_{s-1}$ for $r+1 \le s \le m$. We write $\overline{\sigma_0} = \begin{pmatrix} x_0 & x_1 & \cdots\cdots x_m \\ x_0 & x_1 & \cdots\cdots x_m \end{pmatrix}$ and

$$\overline{\sigma_i} = \begin{pmatrix} x_0 & x_1 & \cdots\cdots x_i & x_{i+1} \cdots\cdots x_m \\ x_0 & x_0 & \cdots\cdots x_0 & x_1 \cdots\cdots x_{m-1} \end{pmatrix}, \quad 1 \le i \le m.$$

Then $End\, f = \overline{E(m)} = \{\overline{\sigma_0}, \ \overline{\sigma_1}, \cdots\cdots, \overline{\sigma_m}\}$ with multiplication given by $\overline{\sigma_0}\,\overline{\sigma_r} = \overline{\sigma_r}\,\overline{\sigma_0} = \overline{\sigma_r}$ for $0 \le r \le m$, $\overline{\sigma_i}\,\overline{\sigma_j} = \overline{\sigma_j}\,\overline{\sigma_i} = \overline{\sigma_{i+j}}$ for $0 < i, j < m$ and $i + j \le m$, and $\overline{\sigma_r}\,\overline{\sigma_m} = \overline{\sigma_m}\,\overline{\sigma_r} = \overline{\sigma_m}$ for $0 \le r \le m$. Then it is clear that $\overline{E(m)} \cong E(m)$ and hence statement of the lemma is alright.

We next consider the following situation:

Lemma 3.2 Let f be given by the directed graph $G(f)$ in fig.3:

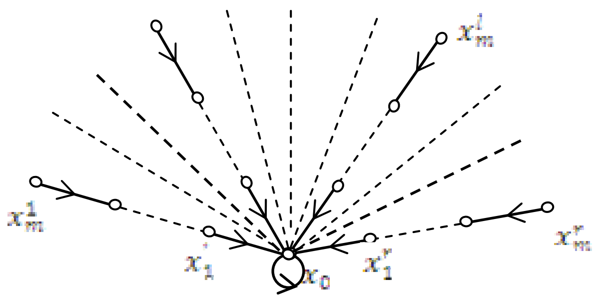

Fig.3

consisting of *r* directed subgraphs each being a chain of length *m* and each with the loops at x_0. Then $End\, f \cong E(m) \varsigma F_r$ -------- (1).

Here, as mentioned earlier, F_r is the full transformation semigroup on a set with *r* elements.

Proof. Since each $g \in End\, f$ must map x_0 onto itself and since each maximal chain ending at x_0 has the same length m, $End\, f$ may be identified with the semigroup of all endomorphisms of an endomapping f' of X whose directed graph is $C_m \times \{1, \ 2, \cdots\cdots, r\}$, C_m being the directed graph given by the figure in Lemma 3.3. It therefore follows from the mentioned lemma and definition of wreath product that $End\, f \cong E(m)\, \varsigma F_r$.

We now observe that if *f* is given by the directed graph *G*(*f*) in fig.4:

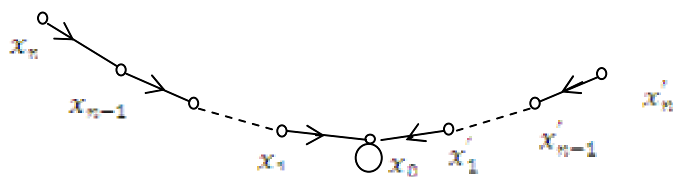

Fig.4

with m > n , then

(2) $$\begin{cases} End\ f = (End\ f_1 \times End\ f_2) \cup (End\ f_1 \times Hom(T_2, T_1)) \cup \\ \qquad (Hom(T_1, T_2) \times Hom(T_1 \times T_2)) \cup (Hom(T_1, T_2), End\ f_2), \end{cases}$$

where f_1 and f_2 are f restricted to $\{x_0, x_1, \cdots\cdots, x_m\}$ and $\{x_0, x_1', \cdots\cdots, x_n'\}$ respectively and $Hom(T_i, T_j)$ $(i, j = 1, 2;\ i \neq j)$ denotes the set of maps the directed graph T_i into the directed graph T_j of f_i and f_j respectively.

Here

(3) $$\begin{cases} (End\ T_i)\ Hom(T_j, T_i) \subseteq Hom(T_j, T_i) \\ Hom(T_i, T_j)\ Hom(T_j, T_i) \subseteq End\ T_j. \end{cases}$$

Also, if $\phi = \begin{pmatrix} x_0 & x_1 & \cdots\cdots & x_{m-n} & x_{m-n+1} & \cdots\cdots & x_n \\ x_0 & x_0 & \cdots\cdots & x_0 & x_1' & \cdots\cdots & x_n' \end{pmatrix}$ and $\varphi = \begin{pmatrix} x_0 & x_1' & \cdots\cdots & x_n' \\ x_0 & x_1 & \cdots\cdots & x_n \end{pmatrix}$, then it is easy to see that

(4) $$\begin{cases} Hom(T_1, T_2) = (End\ T_2)\phi \\ Hom(T_2, T_1) = (End\ T_1)\varphi\ . \end{cases}$$

It follows from the above facts that

(5) $$\begin{cases} (End\ f_1) \times Hom(T_2, T_1) = (End\ f_1)^2 \varphi, \\ (Hom(T_1, T_2)(End\ f_1) = (End\ f_2)\phi(End\ f_1), \\ (End\ f_2) \times Hom(T_1, T_2) = (End\ f_2)^2 \varphi, \\ (Hom(T_2, T_1)(End\ f_2) = (End\ f_1)\varphi(End\ f_2), \\ (Hom(T_1, T_2))(Hom(T_2, T_1)) = (End\ f_2)\phi(End\ f_1)\varphi, \\ (Hom(T_2, T_1))(Hom(T_1, T_2)) = (End\ f_1)\varphi(End\ f_2)\phi\ . \end{cases}$$

We therefore have proved that.

Lemma 3.3 The semigroup-structure of $End\ f$ of f given by fig.4 is completely given by the expressions from (1) to (5).

We now consider the following situation:

Let $T_{1,k_1}', \cdots, T_{r_1,k_1}'$ represent r_1 chains, each of length k_1 and each with a loop at the same point x_0. Let T denote the directed graph consisting of all such $T_{1,k_j}', \cdots, T_{r_j,k_j}'$, $1 \leq j \leq n$. Let f be given by the directed graph $G(f) = T$. Then $G(f) = T$ will look like fig.5.

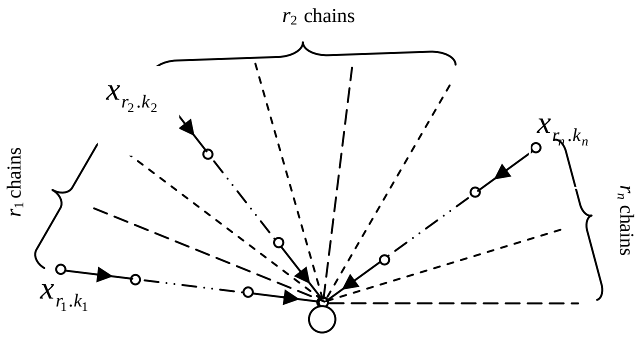

Fig. 3.

Theorem 3.1 Let *f* be given by the directed graph $G(f)$ as in fig.5 with $f^k(X)$ a singleton, where $k = \max\{k_1, \cdots, k_m\}$. Then

(6) $$End\ f = \bigcup [End\ T^{i_1} \times \cdots\cdots \times End\ T^{i_u} \times (\times_{u<v,\ v'<k}^{v\neq v'} Hom\ (T^{i_v}, T^{i_{v'}}))],$$

the union being taken over all permutations $\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots k \\ i_1 & i_2 & i_3 & \cdots\cdots i_k \end{pmatrix}$.

Here,

(7) $$\begin{cases} End\,T^i = End\,T^{i,\alpha} \varsigma F_{i_r} \quad (\alpha \in \{1, \ 2, \ 3, \cdots\cdots, \ i_r\}), \\ Hom(T^{i_v}, T^{i_{v'}}) = \times_{1\leq \beta \leq r_{i_{v'}}}^{1\leq \alpha \leq r_{i_v}} Hom(T_\alpha^{i_v}, T_\beta^{i_{v'}}). \end{cases}$$

The products of $End\ T^{i,\alpha}$ with themselves and with $Hom(T_\alpha^i, T_\beta^j)$ as well as the products of $Hom(T_\alpha^i, T_\beta^j)$ among themselves are given by (5). Also, the $End\ T^{i,\alpha}$'s are isomorphic to one another, since $T^{i,\alpha}$'s are chains of the same length.

Proof. The proof is exactly similar to that of lemma 3.3.

Conclusion

In the most general case of an endomapping, the directed graph representing the endomapping consists of a finite number of disjoint directed rooted trees. In this case, the method of determining the endomorphism semigroup of this endomapping will be almost similar but complicated. It will be taken up in future.

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