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HARMONIC BALANCE SOLUTION OF COUPLED NONLINEAR NON-CONSERVATIVE DIFFERENTIAL EQUATION

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ABSTRACT

A modified harmonic balance method is employed to determine the second approximate solutions to a coupled nonlinear differential equation near the limit cycle. The solution shows a good agreement with the numerical solution.

Keywords: Harmonic balance method, Coupled van der Pol equation, Nonlinear oscillator

1. Introduction

The system of two equations frequently arises in nonlinear oscillations, nonlinear dynamics and mathematical physics etc. Rand and Holmes [1] first proposed the system of two coupled van der Pol equations with linear diffusive coupling. They investigated the properties of certain periodic motion of two identical van der Pol oscillators with weak nonlinear coupling. Recently, Naeem et. al. [2] has found approximate first integral for a system of two coupled van der Pol oscillators with linear diffusive coupling.

There are many analytical approaches for approximating periodic solutions of the nonlinear systems. The most widely used methods are the perturbation methods, in which the solution is expanded in power series of a small parameter. The LP method [3], KBM method [4-5] and multi-time expansion method [6-7] are important among them. Usually, a lower order (*e.g.*, first or second) approximate solution is determined by the perturbation methods due to avoid algebraic complexities. To tackle similar nonlinear problems, there are more important approximation techniques. One of them is the iteration technique (see [8-9]).

The harmonic balance (HB) method [10-19] is another technique for determining periodic solutions of nonlinear differential equations by using the truncated Fourier series. Since the derivation of higher approximation is complicated, the first and second approximate solutions are usually calculated. The advantage of HB method is that the solution gives desire result though nonlinearities become significant.

The aim of this article is to find a second approximation to a coupled van der Polûs equation following a modified harmonic balance technique.

2. The solution method

Let us consider a nonlinear coupled differential equation

$$\ddot{u}+u-\varepsilon f(U,\dot{u})=\varepsilon(Au+B\dot{u}), \qquad \varepsilon<<1 \tag{1}$$

where $u=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $U=\begin{pmatrix}x_1 & 0\\0 & x_2\end{pmatrix}$,

and

$$A=\alpha\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & -\Delta\end{bmatrix},\ B=\beta\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}.$$

A periodic solution of Eq. (1) is chosen in the form

$$\begin{aligned}x_1&=a\cos\varphi+a^3(c_3\cos3\varphi+d_3\sin3\varphi)+a^5(c_5\cos5\varphi+d_5\sin5\varphi)+\cdots\\x_2&=b\cos\psi+b^3(p_3\cos3\psi+q_3\sin3\psi)+b^5(p_5\cos5\psi+q_5\sin5\psi)+\cdots\end{aligned} \tag{2}$$

where a, $\dot{\varphi}$ and $\dot{\psi}$ are constants. In general the unknown functions, $c_j(a)$, $d_j(a)$, $p_j(a)$, $q_j(a)$, $j=3,5,\cdots$ are determined together with a and the initial phase, φ_0 and ψ_0.

Now substituting Eq. (2) into Eq. (1) and expanding the function $f(U,\dot{u})$ in a Fourier series, we obtain

$$\begin{aligned}&a(1-\dot{\varphi}^2)\cos\varphi+a^3c_3(1-9\dot{\varphi}^2)\cos3\varphi+a^3d_3(1-9\dot{\varphi}^2)\sin3\varphi+\cdots\\&-\varepsilon\big[aF_1(a,c_3,d_3,p_3,q_3\cdots)\cos\varphi+a^3F_3(a,c_3,d_3,p_3,q_3\cdots)\cos3\varphi+\cdots\\&+aG_1(a,c_3,d_3,p_3,q_3\cdots)\sin\varphi+a^3G_3(a,c_3,d_3,p_3,q_3\cdots)\sin3\varphi\cdots\\&+bf_1(a,c_3,d_3,p_3,q_3\cdots)\cos\psi+b^3f_3(a,c_3,d_3,p_3,q_3\cdots)\sin3\psi\cdots\\&+bg_1(a,c_3,d_3,p_3,q_3\cdots)\sin\psi+b^3g_3(a,c_3,d_3,p_3,q_3\cdots)\sin3\psi\cdots\ \big]\\&=-\varepsilon aA\cos\varphi-\varepsilon a^3(Ac_3+3Bd_3\dot{\varphi})\cos3\varphi+\cdots\\&+\varepsilon aB\dot{\varphi}\sin\varphi+\varepsilon a^3(3Bc_3\dot{\varphi}-Ad_3)\sin3\varphi+\cdots\\&+\varepsilon Ab\cos\psi+\varepsilon b^3(Ap_3+3Bq_3\dot{\psi})\cos3\psi+\cdots\\&-\varepsilon bB\dot{\psi}\sin\psi+\varepsilon b^3(-3Bp_3\dot{\psi}+Aq_3)\sin3\psi+\cdots\end{aligned} \tag{3}$$

and

$$\begin{aligned}&b(1-\dot{\psi}^2)\cos\psi+b^3p_3(1-9\dot{\psi}^2)\cos3\psi+b^3q_3(1-9\dot{\psi}^2)\sin3\psi+\cdots\\&-\varepsilon\big[aF_1(a,c_3,d_3,p_3,q_3\cdots)\cos\varphi+a^3F_3(a,c_3,d_3,p_3,q_3\cdots)\cos3\varphi+\cdots\end{aligned} \tag{4}$$

$$+aG_1(a,c_3,d_3,p_3,q_3\cdots)\sin\varphi+a^3G_3(a,c_3,d_3,p_3,q_3\cdots)\sin 3\varphi\cdots$$
$$+bf_1(a,c_3,d_3,p_3,q_3\cdots)\cos\psi+b^3f_3(a,c_3,d_3,p_3,q_3\cdots)\sin 3\psi\cdots$$
$$+bg_1(a,c_3,d_3,p_3,q_3\cdots)\sin\psi+b^3g_3(a,c_3,d_3,p_3,q_3\cdots)\sin 3\psi\cdots\ ]$$
$$=\varepsilon a A\cos\varphi+\varepsilon a^3(Ac_3+3Bd_3\dot\varphi)\cos 3\varphi+\cdots$$
$$-\varepsilon a B\dot\varphi\sin\varphi+\varepsilon a^3(-3Bc_3\dot\varphi+Ad_3)\sin 3\varphi+\cdots$$
$$-\varepsilon b(A+\Delta)\cos\psi-\varepsilon b^3(Ap_3+3Bq_3\dot\psi+\Delta p_3)\cos 3\psi+\cdots$$
$$+\varepsilon bB\dot\psi\sin\psi+\varepsilon b^3(3Bp_3\dot\psi-Aq_3-\Delta q_3)\sin 3\psi+\cdots$$

By comparing the coefficients of equal harmonic from Eq. (3) and Eq. (4), we obtain

$$a(1-\dot\varphi^2)-\varepsilon aF_1=-\varepsilon aA,\quad a^3c_3(1-9\dot\varphi^2)-\varepsilon a^3F_3=-\varepsilon a^3(Ac_3+3Bd_3\dot\varphi)\qquad(5)$$
$$\varepsilon aG_1=-\varepsilon aB\dot\varphi,\ a^3d_3(1-9\dot\varphi^2)-\varepsilon a^3G_3=\varepsilon a^3(3Bc_3\dot\varphi-Ad_3)$$

and

$$b(1-\dot\psi^2)-\varepsilon bf_1=-\varepsilon b(A+\Delta),$$
$$b^3p_3(1-9\dot\psi^2)-\varepsilon b^3f_3=-\varepsilon b^3(Ap_3+3Bq_3\dot\psi+\Delta p_3)$$
$$\varepsilon bg_1=-\varepsilon bB\dot\psi,\ b^3q_3(1-9\dot\psi^2)-\varepsilon b^3g_3=\varepsilon b^3(3Bp_3\dot\psi-Aq_3-\Delta q_3)\qquad(6)$$

Utilizing the first equation of Eq. (5), we eliminate $\dot\varphi^2$ from all the rest. Thus Eq.(5) takes the following form

$$\dot\varphi^2=1+\varepsilon A-\varepsilon F_1$$
$$G_1=-B\dot\varphi$$
$$c_3=-\{9(\varepsilon A-\varepsilon F_1)c_3+\varepsilon F_3-\varepsilon(Ac_3+3Bd_3\dot\varphi)\}/8\qquad(7)$$
$$d_3=-\{9(\varepsilon A-\varepsilon F_1)d_3+\varepsilon G_3+\varepsilon(3Bc_3\dot\varphi-Ad_3)\}/8$$

Again utilizing the first equation of Eq. (6), we eliminate $\dot\psi^2$ from all the rest. Thus Eq.(6) takes the following form

$$\dot\psi^2=1-\varepsilon f_1+\varepsilon(A+\Delta)$$
$$g_1=-B\dot\psi$$
$$p_3=-\{9(-\varepsilon f_1+\varepsilon(A+\Delta))p_3+\varepsilon f_3-\varepsilon(Ap_3+3Bq_3\dot\psi+\Delta p_3)\}/8$$
$$q_3=-\{9(-\varepsilon f_1+\varepsilon(A+\Delta))q_3+\varepsilon g_3+\varepsilon(3Bp_3\dot\psi-Aq_3-\Delta q_3)\}/8\qquad(8)$$

We use new parameter $\mu(\varepsilon,\dot\varphi)<<1$ and $\gamma(\varepsilon,\dot\psi)<<1$ with $\varepsilon=O(1)$ and solve the third-, fourth-etc. equations of Eq. (7) and Eq. (8) in powers of μ and γ respectively as

$$c_j = c_{j,1}\mu + c_{j,2}\mu^2 + c_{j,3}\mu^3 + \cdots\cdots \quad j = 3,5,\cdots$$
$$d_j = d_{j,1}\mu + d_{j,2}\mu^2 + d_{j,3}\mu^3 + \cdots\cdots \tag{9}$$

and

$$p_j = p_{j,1}\gamma + p_{j,2}\gamma^2 + p_{j,3}\gamma^3 + \cdots\cdots \quad j = 3,5,\cdots$$
$$q_j = q_{j,1}\gamma + q_{j,2}\gamma^2 + q_{j,3}\gamma^3 + \cdots\cdots \tag{10}$$

Now substituting the values of $c_3, c_5, \cdots, d_3, d_5 \cdots$ and $p_3, p_5, \cdots, q_3, q_5 \cdots$ from Eq. (9) and Eq. (10) respectively into the first equation of Eq. (7) and Eq. (8) respectively, we determine $\dot{\varphi}$ and $\dot{\psi}$, and then φ and ψ.

To determine a steady state solution we start from $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$. Thus we obtain

$$a\sin\varphi_0 + 3a^3(c_3 \sin 3\varphi_0 - d_3 \cos 3\varphi_0) + \cdots = 0 \tag{11}$$
$$b\sin\psi_0 + 3b^3(p_3 \sin 3\psi_0 - q_3 \cos 3\psi_0) + \cdots = 0$$

where $\varphi(0) = \varphi_0$ and $\psi(0) = \psi_0$.

Finally, substituting the values of $c_3, c_5, d_3, d_5, \cdots$ and $p_3, p_5, q_3, q_5, \cdots$ into second equation of Eq. (7), Eq. (8) and Eq. (11), we solve them for a, b, φ_0 and ψ_0. Then substituting the values of a, b, φ_0 and ψ_0 into the equation,

$$x_1(0) = a_0 = a\cos\varphi_0 + a^3 c_3 \cos 3\varphi_0 + a^3 d_3 \sin 3\varphi_0 + \cdots$$
$$x_2(0) = b_0 = b\cos\psi_0 + b^3 p_3 \cos 3\psi_0 + b^3 q_3 \sin 3\psi_0 + \cdots \tag{12}$$

We obtain the value of a_0 and b_0, which represents the initial value of x_1 and x_2 for the steady-state solution.

3. Example

Consider a coupled van der Pol equation

$$\ddot{u} + u - \varepsilon(1 - U^2)\dot{u} = \varepsilon(Au + B\dot{u}), \ \varepsilon << 1 \tag{13}$$

where $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $U = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$

and

$$A = \alpha\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\Delta \end{bmatrix}, \ B = \beta\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let us consider a periodic solution of the form

$$x_1 = a\cos\varphi + a^3(c_3\cos 3\varphi + d_3\sin 3\varphi) + a^5(c_5\cos 5\varphi + d_5\sin 5\varphi)$$
$$x_2 = b\cos\psi + b^3(p_3\cos 3\psi + q_3\sin 3\varphi) + b^5(p_5\cos 5\varphi + q_5\sin 5\varphi) \tag{14}$$

Substituting Eq. (14) into Eq. (13), we get

$$\begin{aligned}
&a(1-\dot\varphi^2)\cos\varphi + a^3c_3(1-9\dot\varphi^2)\cos 3\varphi + a^3d_3(1-9\dot\varphi^2)\sin 3\varphi \\
&+ a^5c_5(1-25\dot\varphi^2)\cos 5\varphi + a^5d_5(1-25\dot\varphi^2)\sin 5\varphi - \frac{\varepsilon}{4}[(-4a + a^3 + a^5c_3 \\
&+ 2a^7{c_3}^2 + 2a^7{d_3}^2 + 2a^{11}{c_5}^2 + 2a^{11}{d_5}^2 + 2a^9c_3c_5 + 2a^9d_3d_5 + 2a^{11}c_3d_3d_5 \\
&+ a^{11}{c_3}^2c_5 - a^{11}{d_3}^2c_5)\sin\varphi + (a^3 - 12a^3c_3 + 6a^5c_3 + 6a^9c_3c_5 + 6a^9d_3d_5 + 3a^7c_5 \\
&+ 3a^9{c_3}^3 + 3a^9c_3{d_3}^2 + 6a^{13}c_3{c_5}^2 + 6a^{13}c_3{d_5}^2)\sin 3\varphi + (-20a^5c_5 + 5a^5c_3 + 5a^7{c_3}^2 \\
&- 5a^7{d_3}^2 + 10a^7c_5 + 10a^{11}{c_3}^2c_5 + 10a^{11}c_5{d_3}^2 + 5a^{15}{c_5}^3 + 5a^{15}c_5{d_5}^2)\sin 5\varphi \\
&- (a^5d_3 + 2a^{11}c_3c_5d_3 + a^{11}{d_3}^2d_5 - 2a^9c_5d_3 + 2a^9c_3d_5 - a^{11}{c_3}^2d_5)\cos\varphi \\
&+ (12a^3d_3 - 6a^5d_3 - 3a^9{c_3}^2d_3 - 3a^9{d_3}^3 - 6a^{13}{c_5}^2d_3 - 6a^9d_3{d_5}^2 - 3a^7d_5 \\
&- 6a^9c_3d_5 + 6a^9c_5d_3)\cos 3\varphi + (20a^5d_5 - 5a^5d_3 - 10a^7d_5 - 10a^7c_3d_3 \\
&- 10a^{11}{c_3}^2d_5 - 10a^{11}{d_3}^2d_5 - 5a^{15}{c_5}^2d_5 - 5a^{15}{d_5}^3)\cos 5\varphi]\dot\varphi + \text{HOH} \\
&= \varepsilon[aB\dot\varphi\sin\varphi + a^3(3Bc_3\dot\varphi - Ad_3)\sin 3\varphi + a^5(5Bc_5\dot\varphi - Ad_5)\sin 5\varphi \\
&- aA\cos\varphi - a^3(Ac_3 + 3Bd_3\dot\varphi)\cos 3\varphi - a^5(Ac_5 + 5Bd_5\dot\varphi)\cos 5\varphi] + \text{HOH}
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&b(1-\dot\psi^2)\cos\psi + b^3p_3(1-9\dot\psi^2)\cos 3\psi + b^3q_3(1-9\dot\psi^2)\sin 3\psi \\
&+ b^5p_5(1-25\dot\psi^2)\cos 5\psi + b^5q_5(1-25\dot\psi^2)\sin 5\psi - \frac{\varepsilon}{4}[(-4b + b^3 + b^5p_3 \\
&+ 2b^7{p_3}^2 + 2b^7{q_3}^2 + 2b^{11}{p_5}^2 + 2b^{11}{q_5}^2 + 2b^9p_3p_5 + 2b^9q_3q_5 + 2b^{11}p_3q_3q_5 \\
&+ b^{11}{p_3}^2p_5 - b^{11}{q_3}^2p_5)\sin\psi + (b^3 - 12b^3p_3 + 6b^5p_3 + 6b^9p_3p_5 + 6b^9q_3q_5 + 3b^7p_5 \\
&+ 3b^9{p_3}^3 + 3b^9p_3{q_3}^2 + 6b^{13}p_3{p_5}^2 + 6b^{13}p_3{q_5}^2)\sin 3\psi + (-20b^5p_5 + 5b^5p_3 + 5b^7p_3^2 \\
&- 5b^7{q_3}^2 + 10b^7p_5 + 10b^{11}{p_3}^2p_5 + 10b^{11}p_5{q_3}^2 + 5b^{15}{p_5}^3 + 5b^{15}p_5{q_5}^2)\sin 5\psi \\
&- (b^5q_3 + 2b^{11}p_3p_5q_3 + b^{11}{q_3}^2q_5 - 2b^9p_5q_3 + 2b^9p_3q_5 - b^{11}{p_3}^2q_5)\cos\psi \\
&+ (12b^3q_3 - 6b^5q_3 - 3b^9{p_3}^2q_3 - 3b^9{q_3}^3 - 6b^{13}{p_5}^2q_3 - 6b^9q_3{q_5}^2 - 3b^7q_5
\end{aligned} \tag{16}$$

$$-6b^9 p_3 q_5 + 6b^9 p_5 q_3)\cos 3\psi + (20b^5 q_5 - 5b^5 q_3 - 10b^7 q_5 - 10b^7 p_3 q_3$$

$$-10b^{11} p_3{}^2 q_5 - 10b^{11} q_3{}^2 q_5 - 5b^{15} p_5{}^2 q_5 - 5b^{15} q_5{}^3)\cos 5\psi]\dot{\psi} + \text{HOH}$$

$$= \varepsilon[bB\dot{\psi}\sin\psi + b^3(3B\,p_3\dot{\psi} - A\,q_3 - \Delta\,q_3)\sin 3\psi + b^5(5B\,p_5\dot{\psi} - A\,q_5 - \Delta\,q_5)\sin 5\psi$$

$$-b(A+\Delta)\cos\psi - b^3(A\,p_3 + 3B\,q_3\dot{\psi} + \Delta\,p_3)\cos 3\psi$$

$$-b^5(A\,p_5 + 5B\,q_5\dot{\psi} + \Delta\,p_5)\cos 5\psi] + \text{HOH}$$

where HOH stands for the higher order harmonics.

Comparing the coefficients of equal harmonics, we obtain

$$\dot{\varphi}^2 = 1 + \varepsilon\,A + \frac{\varepsilon\dot{\varphi}}{4}(a^4 d_3 - 2a^8 c_5 d_3 + 2a^8 c_3 d_5 + 2a^{10} c_3 c_5 d_3 + a^{10} d_3{}^2 d_5 - a^{10} c_3{}^2 d_5)$$

$$c_3(1 - 9\dot{\varphi}^2) = -\varepsilon\,A c_3 + \frac{\varepsilon\dot{\varphi}}{4}(12d_3 - 6a^2 d_3 - 3a^4 d_5 - 6a^6 c_3 d_5 + 6a^6 c_5 d_3 - 3a^6 c_3{}^2 d_3$$
$$- 3a^6 d_3{}^3 - 6a^{10} c_5{}^2 d_3 - 6a^6 d_3 d_5{}^2 - 12B d_3)$$

$$c_5(1 - 25\dot{\varphi}^2) = -\varepsilon\,A c_5 + \frac{\varepsilon\dot{\varphi}}{4}(20d_5 - 5d_3 - 10a^2 d_5 - 10a^2 c_3 d_3 - 10a^6 c_3{}^2 d_5$$
$$-10a^6 d_3{}^2 d_5 - 5a^{10} c_5{}^2 d_5 - 5a^{10} d_5{}^3 - 20B\,d_5) \quad (17)$$

$$4 - a^2 - 4B - a^4 c_3 - 2a^6 c_3{}^2 - 2a^6 d_3{}^2 - 2a^{10} c_5{}^2 - 2a^{10} d_5{}^2$$
$$-2a^8 c_3 c_5 - 2a^8 d_3 d_5 - 2a^{10} c_3 d_3 d_5 - a^{10} c_3{}^2 c_5 + a^{10} d_3{}^2 c_5 = 0$$

$$d_3(1 - 9\dot{\varphi}^2) = -\varepsilon\,A d_3 + \frac{\varepsilon\dot{\varphi}}{4}(1 - 12c_3 + 6a^2 c_3 + 3a^4 c_5 + 6a^6 c_3 c_5 + 6a^6 d_3 d_5$$
$$+ 3a^6 c_3{}^3 + 3a^6 c_3 d_3{}^2 + 6a^{10} c_3 c_5{}^2 + 6a^{10} c_3 d_5{}^2 + 12B\,c_3)$$

$$d_5(1 - 25\dot{\varphi}^2) = -\varepsilon\,A d_5 + \frac{\varepsilon\dot{\varphi}}{4}(-20c_5 + 5c_3 + 5a^2 c_3{}^2 - 5a^2 d_3{}^2 + 10a^2 c_5$$
$$+10a^6 c_3{}^2 c_5 + 10a^6 c_5 d_3{}^2 + 5a^{10} c_5{}^3 + 5a^{10} c_5 d_5{}^2 + 20B\,c_5)$$

and

$$\dot{\psi}^2 = 1 + \varepsilon\,A + \varepsilon\,\Delta + \frac{\varepsilon\dot{\psi}}{4}(b^4 q_3 - 2b^8 p_5 q_3 + 2b^8 p_3 q_5$$
$$+ 2b^{10} p_3 p_5 q_3 + b^{10} q_3{}^2 q_5 - b^{10} p_3{}^2 q_5)$$

$$p_3(1 - 9\dot{\psi}^2) = -\varepsilon\,A\,p_3 - \varepsilon\,\Delta\,p_3 + \frac{\varepsilon\dot{\psi}}{4}(12q_3 - 6b^2 q_3 - 3b^4 q_5 - 6b^6 p_3 q_5 + 6b^6 p_5 q_3$$
$$-3b^6 p_3{}^2 q_3 - 3b^6 q_3{}^3 - 6b^{10} p_5{}^2 q_3 - 6b^6 q_3 q_5{}^2 - 12B q_3)$$

$$p_5(1-25\dot{\psi}^2) = -\varepsilon A\, p_5 - \varepsilon\,\Delta\, p_5 + \frac{\varepsilon\dot{\psi}}{4}(20q_5 - 5q_3 - 10b^2 q_5 - 10b^2 p_3 q_3 - 10b^6 p_3{}^2 q_5 - 10b^6 q_3{}^2 q_5 - 5b^{10} p_5{}^2 q_5 - 5b^{10} q_5{}^3 - 20B\, q_5) \quad (18)$$

$$4 - b^2 - 4B - b^4 p_3 - 2b^6 p_3{}^2 - 2b^6 q_3{}^2 - 2b^{10} p_5{}^2 - 2b^{10} q_5{}^2 - 2b^8 p_3 p_5 - 2b^8 q_3 q_5 - 2b^{10} p_3 q_3 q_5 - b^{10} p_3{}^2 p_5 + b^{10} q_3{}^2 p_5 = 0$$

$$q_3(1-9\dot{\psi}^2) = -\varepsilon A\, q_3 - \varepsilon\,\Delta\, q_3 + \frac{\varepsilon\dot{\psi}}{4}(1 - 12p_3 + 6b^2 p_3 + 3b^4 p_5 + 6b^6 p_3 p_5 + 6b^6 q_3 q_5 + 3b^6 p_3{}^3 + 3b^6 p_3 q_3{}^2 + 6b^{10} p_3 p_5{}^2 + 6b^{10} p_3 q_5{}^2 + 12B\, p_3)$$

$$q_5(1-25\dot{\psi}^2) = -\varepsilon A\, q_5 - \varepsilon\,\Delta\, q_5 + \frac{\varepsilon\dot{\psi}}{4}(-20p_5 + 5p_3 + 5b^2 p_3{}^2 - 5b^2 q_3{}^2 + 10b^2 p_5 + 10b^6 p_3{}^2 p_5 + 10b^6 p_5 q_3{}^2 + 5b^{10} p_5{}^3 + 5b^{10} p_5 q_5{}^2 + 20B\, p_5)$$

With the help of the first equation of Eq. (17) and Eq. (18), we eliminate $\dot{\varphi}^2$ and $\dot{\psi}^2$ respectively from the second, third, fifth and sixth equation of Eq. (19) and Eq. (20), we obtain

$$4 - a^2 - 4B - a^4 c_3 - 2a^6 c_3{}^2 - 2a^6 d_3{}^2 - 2a^{10} c_5{}^2 - 2a^{10} d_5{}^2 - 2a^8 c_3 c_5 - 2a^8 d_3 d_5 - 2a^{10} c_3 d_3 d_5 - a^{10} c_3{}^2 c_5 + a^{10} d_3{}^2 c_5 = 0$$

$$\dot{\varphi}^2 = 1 + \varepsilon A + \frac{\varepsilon\dot{\varphi}}{4}(a^4 d_3 - 2a^8 c_5 d_3 + 2a^8 c_3 d_5 + 2a^{10} c_3 c_5 d_3 + a^{10} d_3{}^2 d_5 - a^{10} c_3{}^2 d_5)$$

$$c_3 = \frac{\varepsilon\dot{\varphi}}{96(1+\varepsilon A)}(-36d_3 + 18a^2 d_3 + 36Bd_3 - 27a^4 c_3 d_3 + 9a^6 c_3{}^2 d_3 - 18a^6 c_5 d_3 + 54a^8 c_3 c_5 d_3 - 54a^{10} c_3{}^2 c_5 d_3 + 18a^{10} c_5{}^2 d_3 + 9a^6 d_3{}^3 + 9a^4 d_5 + 18a^6 c_3 d_5 - 54a^8 c_3{}^2 d_5 + 27a^{10} c_3{}^3 d_5 - 27a^{10} c_3 d_3{}^2 d_5 + 18a^{10} d_3 d_5{}^2)$$

$$c_5 = \frac{\varepsilon\dot{\varphi}}{96(1+\varepsilon A)}(5d_3 + 10a^2 c_3 d_3 - 25a^4 c_5 d_3 + 50a^8 c_5{}^2 d_3 - 50a^{10} c_3 c_5{}^2 d_3 - 20d_5 + 10a^2 d_5 + 20B\, d_5 + 10a^6 c_3{}^2 d_5 - 50a^8 c_3 c_5 d_5 + 25a^{10} c_3{}^2 c_5 d_5 + 5a^{10} c_5{}^2 d_5 + 10a^6 d_3{}^2 d_5 - 25a^{10} c_5 d_3{}^2 d_5 + 5a^{10} d_5{}^3) \quad (19)$$

$$d_3 = \frac{\varepsilon\dot{\varphi}}{96(1+\varepsilon A)}(-3 + 36c_3 - 18a^2 c_3 - 36B\, c_3 - 9a^6 c_3{}^3 - 9a^4 c_5 - 18a^6 c_3 c_5 - 18a^{10} c_3 c_5{}^2 - 27a^4 d_3{}^2 - 9a^6 c_3 d_3{}^2 + 54a^8 c_5 d_3{}^2 - 54a^{10} c_3 c_5 d_3{}^2 - 18a^6 d_3 d_5 - 54a^8 c_3 d_3 d_5 + 27a^{10} c_3{}^2 d_3 d_5 - 27a^{10} d_3{}^3 d_5 - 18a^{10} c_3 d_5{}^2)$$

$$d_5 = \frac{\varepsilon\dot{\varphi}}{96(1+\varepsilon A)}(-5c_3 - 5a^2 c_3{}^2 + 20c_5 - 10a^2 c_5 - 20B\, c_5 - 10a^6 c_3{}^2 c_5$$
$$- 5a^{10} c_5{}^3 + 5a^2 d_3{}^2 - 10a^6 c_5 d_3{}^2 - 25a^4 d_3 d_3 + 50a^8 c_5 d_3 d_5$$
$$- 50a^{10} c_3 c_5 d_3 d_5 - 50a^8 c_3 d_5{}^2 + 25a^{10} c_3{}^2 d_5{}^2 - 5a^{10} c_5 d_5{}^2 - 25a^{10} d_3{}^2 d_5{}^2)$$

and

$$4 - b^2 - 4B - b^4 p_3 - 2b^6 p_3{}^2 - 2b^6 q_3{}^2 - 2b^{10} p_5{}^2 - 2b^{10} q_5{}^2$$
$$- 2b^8 p_3 p_5 - 2b^8 q_3 q_5 - 2b^{10} p_3 q_3 q_5 - b^{10} p_3{}^2 p_5 + b^{10} q_3{}^2 p_5 = 0$$

$$\dot{\psi}^2 = 1 + \varepsilon A + \varepsilon \Delta + \frac{\varepsilon\dot{\psi}}{4}(b^4 q_3 - 2b^8 p_5 q_3 + 2b^8 p_3 q_5$$
$$+ 2b^{10} p_3 p_5 q_3 + b^{10} q_3{}^2 q_5 - b^{10} p_3{}^2 q_5)$$

$$p_3 = \frac{\varepsilon\dot{\psi}}{96(1+\varepsilon A + \varepsilon\Delta)}(-36 q_3 + 18b^2 q_3 + 36B q_3 - 27b^4 p_3 q_3 + 9b^6 p_3{}^2 q_3$$
$$- 18b^6 p_5 q_3 + 54b^8 p_3 p_5 q_3 - 54b^{10} p_3{}^2 p_5 q_3 + 18b^{10} p_5{}^2 q_3 + 9b^6 q_3{}^3 + 9b^4 q_5$$
$$+ 18b^6 p_3 q_5 - 54b^8 p_3{}^2 q_5 + 27b^{10} p_3{}^3 q_5 - 27b^{10} p_3 q_3{}^2 q_5 + 18b^{10} q_3 q_5{}^2)$$

$$p_5 = \frac{\varepsilon\dot{\psi}}{96(1+\varepsilon A + \varepsilon\Delta)}(5q_3 + 10b^2 p_3 q_3 - 25b^4 p_5 q_3 + 50b^8 p_5{}^2 q_3$$
$$- 50b^{10} p_3 p_5{}^2 q_3 - 20q_5 + 10b^2 q_5 + 20B\, q_5 + 10b^6 p_3{}^2 q_5 - 50b^8 p_3 p_5 q_5 \qquad (20)$$
$$+ 25b^{10} p_3{}^2 p_5 q_5 + 5b^{10} p_5{}^2 q_5 + 10b^6 q_3{}^2 q_5 - 25b^{10} p_5 q_3{}^2 q_5 + 5b^{10} q_5{}^3)$$

$$q_3 = \frac{\varepsilon\dot{\psi}}{96(1+\varepsilon A + \varepsilon\Delta)}(-3 + 36 p_3 - 18b^2 p_3 - 36B\, p_3 - 9b^6 p_3{}^3 - 9b^4 p_5 - 18b^6 p_3 p_5$$
$$- 18b^{10} p_3 p_5{}^2 - 27b^4 q_3{}^2 - 9b^6 p_3 q_3{}^2 + 54b^8 p_5 q_3{}^2 - 54b^{10} p_3 p_5 q_3{}^2$$
$$- 18b^6 q_3 q_5 - 54b^8 p_3 q_3 q_5 + 27b^{10} p_3{}^2 q_3 q_5 - 27b^{10} q_3{}^3 q_5 - 18b^{10} p_3 q_5{}^2)$$

$$q_5 = \frac{\varepsilon\dot{\psi}}{96(1+\varepsilon A + \varepsilon\Delta)}(-5p_3 - 5b^2 p_3{}^2 + 20 p_5 - 10b^2 p_5 - 20B\, p_5 - 10b^6 p_3{}^2 p_5$$
$$- 5b^{10} p_5{}^3 + 5b^2 q_3{}^2 - 10b^6 p_5 q_3{}^2 - 25b^4 q_3 q_5 + 50b^8 p_5 q_3 q_5$$
$$- 50b^{10} p_3 p_5 q_3 q_5 - 50b^8 p_3 q_5{}^2 + 25b^{10} p_3{}^2 q_5{}^2 - 5b^{10} p_5 q_5{}^2 - 25b^{10} q_3{}^2 q_5{}^2)$$

To determine a steady state solution we start from $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$. Thus we obtain

$$a\sin\varphi_0 + 3a^3(c_3\sin 3\varphi_0 - d_3\cos 3\varphi_0) + 5a^5(c_5\sin 5\varphi_0 - d_5\cos 5\varphi_0) = 0$$
$$b\sin\psi_0 + 3b^3(p_3\sin 3\psi_0 - q_3\cos 3\psi_0) + 5b^5(p_5\sin 5\psi_0 - q_5\cos 5\psi_0) = 0 \qquad (21)$$

Herein fourteen unknown quantities $\dot{\varphi},\ \varphi_0,\ a,\ c_3, c_5, d_3, d_5,\ \dot{\psi}, \psi_0,\ b,\ \ p_3, p_5, q_3$ and q_5 will be calculated from fourteen nonlinear equations described by Eq. (10), Eq. (19), Eq. (20) and Eq. (21). Certainly Eq. (19) and Eq. (20) represents a set of nonlinear algebraic equations with small parameter $\mu = \dfrac{\varepsilon\dot{\varphi}}{96(1+\varepsilon A)}$ and $\gamma = \dfrac{\varepsilon\dot{\psi}}{96(1+\varepsilon A+\varepsilon\Delta)}$.

According to [20], we shall be able to find an approximate solution of Eq. (19) and Eq. (20) in the form

$$c_3 = c_{3,1}\mu + c_{3,2}\mu^2 + c_{3,3}\mu^3 + \cdots$$
$$c_5 = c_{5,1}\mu + c_{5,2}\mu^2 + c_{5,3}\mu^3 + \cdots$$
$$d_3 = d_{3,1}\mu + d_{3,2}\mu^2 + d_{3,3}\mu^3 + \cdots \qquad (22)$$
$$d_5 = d_{5,1}\mu + d_{5,2}\mu^2 + d_{5,3}\mu^3 + \cdots$$

and

$$p_3 = p_{3,1}\gamma + p_{3,2}\gamma^2 + p_{3,3}\gamma^3 + \cdots$$
$$p_5 = p_{5,1}\gamma + p_{5,2}\gamma^2 + p_{5,3}\gamma^3 + \cdots$$
$$q_3 = q_{3,1}\gamma + q_{3,2}\gamma^2 + q_{3,3}\gamma^3 + \cdots \qquad (23)$$
$$q_5 = q_{5,1}\gamma + q_{5,2}\gamma^2 + q_{5,3}\gamma^3 + \cdots$$

Substituting Eq. (22) and Eq. (23) into Eq. (19) and Eq. (20) respectively and equating the coefficients of μ and γ on both sides, we will have a system of linear equations of $c_{3,1}, c_{3,2}, \cdots, c_{5,1}, c_{5,2}, \cdots, d_{3,1}, d_{3,2}, \cdots, d_{5,1}, d_{5,2}, \cdots,\ p_{3,1}, p_{3,2}, \cdots,\ \ p_{5,1}, p_{5,2}, \cdots,$ $q_{3,1}, q_{3,2}, \cdots, q_{5,1}, q_{5,2}, \cdots$. Solving these equations, we obtain

$$c_3 = (108 - 54a^2 - 108B)\mu^2 + (-139968 + 209952a^2 - 99900a^4 + 14310a^6 + 419904B$$
$$- 419904a^2B + 99900a^4B - 419904B^2 + 209952a^2B^2 + 139968B^3)\mu^4 + \cdots$$
$$c_5 = -15\mu^2 + (36240 - 40380a^2 + 9465a^4 - 72480B + 40380a^2B + 36240B^2)\mu^4 + \cdots$$
$$d_3 = -3\mu + (3888 - 3888a^2 + 864a^4 - 7776B + 3888a^2B + 3888B^2)\mu^3$$
$$+ (-5038848 + 10077696a^2 - 7071840a^4 + 2021976a^6 - 195183a^8$$
$$+ 20155392B - 30233088a^2B + 14143680a^4B - 2021976a^6B \qquad (24)$$
$$- 30233088B^2 + 30233088a^2B^2 - 7071840a^4B^2 + 20155392B^3$$
$$- 10077696a^2B^3 - 5038848B^4)\mu^5 + \cdots$$

$$d_5 = (-840 + 465a^2 + 840B)\mu^3 + (1424640 - 2394720a^2 + 1204560a^4 - 170475a^6 - 4273920B + 4789440a^2B - 1204560a^4B + 4273920B^2 - 2394720a^2B^2 - 1424640B^4)\mu^5 + \cdots$$

and

$$p_3 = (108 - 54b^2 - 108B)\mu^2 + (-139968 + 209952b^2 - 99900b^4 + 14310b^6 + 419904B - 419904b^2B + 99900b^4B - 419904B^2 + 209952b^2B^2 + 139968B^3)\mu^4 + \cdots$$

$$p_5 = -15\mu^2 + (36240 - 40380b^2 + 9465b^4 - 72480B + 40380b^2B + 36240B^2)\mu^4 + \cdots$$

$$q_3 = -3\mu + (3888 - 3888b^2 + 864b^4 - 7776B + 3888b^2B + 3888B^2)\mu^3 + (-5038848 + 10077696b^2 - 7071840b^4 + 2021976b^6 - 195183b^8 + 20155392B - 30233088b^2B + 14143680b^4B - 2021976b^6B - 30233088B^2 + 30233088b^2B^2 - 7071840b^4B^2 + 20155392B^3 - 10077696b^2B^3 - 5038848B^4)\mu^5 + \cdots \tag{25}$$

$$q_5 = (-840 + 465b^2 + 840B)\mu^3 + (1424640 - 2394720b^2 + 1204560b^4 - 170475b^6 - 4273920B + 4789440b^2B - 1204560b^4B + 4273920B^2 - 2394720b^2B^2 - 1424640B^4)\mu^5 + \cdots$$

Finally, substituting the values of c_3, c_5, d_3, d_5 and p_3, p_5, q_3, q_5 into first equation of Eq. (19), Eq. (20) and Eq. (21), we solve them for a, b, φ_0 and ψ_0. Then substituting the values of a, b, φ_0 and ψ_0 into the equation

$$\begin{aligned} x_1(0) &= a\cos\varphi_0 + a^3c_3\cos 3\varphi_0 + a^3d_3\sin 3\varphi_0 + a^5c_5\cos 5\varphi_0 + a^5d_5\sin 5\varphi_0 \\ x_2(0) &= b\cos\psi_0 + b^3p_3\cos 3\psi_0 + b^3q_3\sin 3\psi_0 + b^5p_5\cos 5\psi_0 + b^5q_5\sin 5\psi_0 \end{aligned} \tag{26}$$

We obtain the value of $x_1(0)$ and $x_2(0)$, which represents the initial value of x_1 and x_2 for the steady-state solution.

4. Results and Discussion

In order to test the accuracy of an approximate solution, some authors [14, 18, 20] compared analytical solutions to those obtained by numerical techniques. We have compared such an approximate solution of Eq. (13) to numerical solution for $\varepsilon = 0.05$ and $\varepsilon = 0.1$. First of all we plot in Fig. 1(a) and Fig. 1(b), the second approximate solution for $\varepsilon = 0.05$, with initial conditions $\left(x_1(0) = 1.8973874,\ \dot{x}_1(0) = 0,\ x_2(0) = 1.8973893,\ \dot{x}_2(0) = 0\right)$, where unknown coefficients $c_3,\ c_5,\ d_3,\ d_5,\ p_3,\ p_5,\ q_3,\ q_5$; amplitudes a, b and initial phases φ_0, ψ_0 are calculated by the first equations of Eq. (19) and Eq. (20) together with Eqs. (21), (24)-(25) and then substituting these values of $c_3,\ c_5,\ d_3,\ d_5,\ p_3,\ p_5,\ q_3,\ q_5, a, b, \varphi_0, \psi_0$ in Eq. (26), we obtain $x_1(0)$ and $x_2(0)$. Then corresponding numerical solution has been computed by *Runge-Kutta* (fourth-order) method. From the figure it is clear that the analytical solution shows a good coincidence with the numerical solution. In Fig. 1(c), we have shown the corresponding phase difference between two oscillators.

In Fig. 2(a) and Fig. 2(b), the second approximate solution of Eq. (13) for $\varepsilon = 0.1$ with the initial conditions ($x_1(0) = 1.8974499,\ \dot{x}_1(0) = 0,\ x_2(0) = 1.8974769,\ \dot{x}_2(0) = 0$), where unknown coefficients are $c_3,\ c_5,\ d_3,\ d_5,\ p_3,\ p_5,\ q_3,\ q_5$; amplitudes a, b and initial phases φ_0, ψ_0 are calculated by the first equations of Eq. (19) and Eq. (20) together with equations (21), (24)-(25) and then substituting these values of $c_3,\ c_5,\ d_3,\ d_5,\ p_3,\ p_5,\ q_3,\ q_5, a, b, \varphi_0, \psi_0$ in Eq. (26), we obtain $x_1(0)$ and $x_2(0)$. Then corresponding numerical solution has been computed by *Runge-Kutta* (fourth-order) method. From the figure it is clear that the analytical solution shows a good coincidence with the numerical solution. In Fig. 2(c), we have shown the corresponding phase difference between two oscillators.

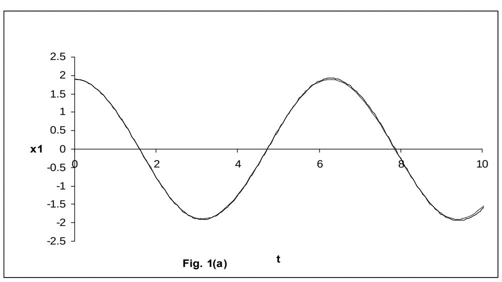


Fig. 1(a)

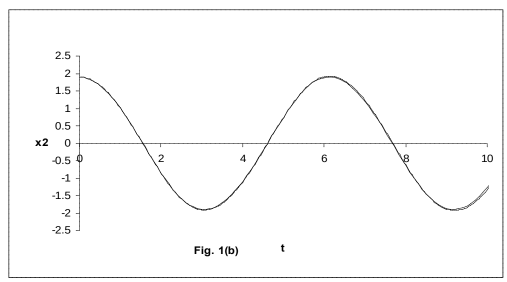


Fig. 1(b)

Fig. 1(a) shows the harmonic balance solution of x_1 of Eq. (13) which is denoted by (é é ..) and the corresponding numerical solution is denoted by (ð). Here $\varepsilon = 0.05$, $A = 0.1$, $B = 0.1$, $\Delta = 1.0$, $a = 1.897397$, $\varphi_0 = -0.017005$, $\dot{\varphi} = 1.0023681$, $c_3 = -0.0000268$, $d_3 = -0.0015738$, $c_5 = -0.0000041$ $d_5 = 0.0000001$and $b = 1.897402$, $\psi_0 = -0.01829$, $\dot{\psi} = 1.0269793$, $p_3 = -0.000031$, $q_3 = -0.0016926$, $p_5 = -0.000004\xi$, $q_5 = 0.0000002$

In Fig. 1(b), we observe the harmonic balance solution of x_2 of Eq. (13) which is denoted by (**….**) and the corresponding numerical solution is denoted by (ð). Here $\varepsilon = 0.05$, $A = 0.1$, $B = 0.1$, $\Delta = 1.0$, $a = 1.897397$, $\varphi_0 = -0.017005$, $\dot{\varphi} = 1.0023681$,

$c_3 = -0.0000268$, $d_3 = -0.0015738$, $c_5 = -0.0000041$, $d_5 = 0.0000001$ and $b = 1.897402$, $\psi_0 = -0.01829$, $\dot{\psi} = 1.0269793$, $p_3 = -0.000031$, $q_3 = -0.0016926$, $p_5 = -0.0000048$, $q_5 = 0.0000002$

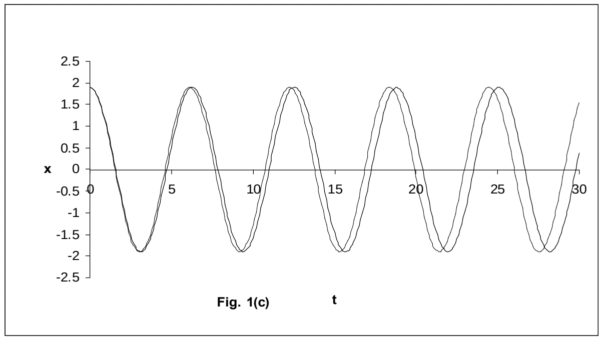
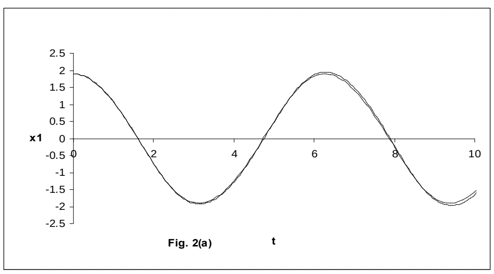


Fig. 1(c)

Fig. 1(c) shows the frequency difference between x_1 and x_2, when $\varepsilon = 0.05$, $A = 0.1$, $B = 0.1$ and $\Delta = 1.0$.



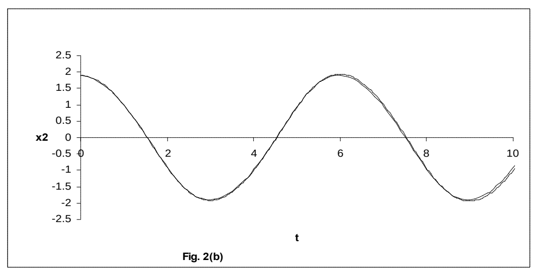


Fig. 2(a) **Fig. 2(b)**

In Fig. 2(a) we observe the harmonic balance solution of x_1 of Eq. (13) which is denoted by (**……..**) and the corresponding numerical solution is denoted by (ð). Here $\varepsilon = 0.1$, $A = 0.1$, $B = 0.1$, $\Delta = 0.5$, $a = 1.89749$, $\varphi_0 = -0.034284$, $\dot{\varphi} = 1.0044638$, $c_3 = -0.0001086$, $d_3 = -0.0031683$, $c_5 = -0.0000167$, $d_5 = 0.0000011$ and $b = 1.897531$, $\psi_0 = -0.039509$, $\dot{\psi} = 1.0528366$, $p_3 = -0.0001441$, $q_3 = -0.0036489$, $p_5 = -0.0000222$, $q_5 = 0.0000017$

In Fig. 2(b) we observe the harmonic balance solution of x_2 of Eq. (13) which is denoted by (**……..**) and the corresponding numerical solution is denoted by (ð). Here $\varepsilon = 0.1$, $A = 0.1$, $B = 0.1$, $\Delta = 0.5$, $a = 1.89749$, $\varphi_0 = -0.034284$, $\dot{\varphi} = 1.0044638$, $c_3 = -0.0001086$, $d_3 = -0.0031683$, $c_5 = -0.0000167$, $d_5 = 0.0000011$ and $b = 1.897531$, $\psi_0 = -0.039509$, $\dot{\psi} = 1.0528366$, $p_3 = -0.0001441$,

$q_3 = -0.0036489$, $p_5 = -0.0000222$, $q_5 = 0.0000017$

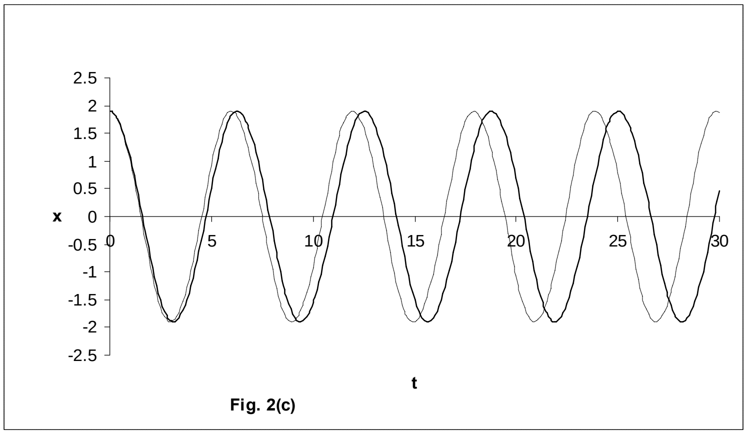


Fig. 2(c)

Fig. 2(c) shows the frequency difference between x_1 and x_2, when $\varepsilon = 0.1$, $A = 0.1$, $B = 0.1$ and $\Delta = 0.5$.

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