GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) Vol. 33 (2013) 25 –32

NUMERICAL SOLUTION OF A MULTILANE TRAFFIC FLOW MODEL

M H Kabir and L S Andallah
Department of Mathematics, Jahangirnagar University, Savar, Dhaka-1342, Bangladesh
Email: hkabirju@gmail.com

Received 12.07.12 Accepted 02.11.13

ABSTRACT

This paper performs the numerical solution of a macroscopic multilane traffic flow model based on a linear density-velocity relationship. A multilane traffic flow is modeled by a system of nonlinear partial differential equation appended with initial and boundary conditions reads as an initial boundary value problem (IBVP). We present numerical simulation of the IBVP by a finite difference scheme named Lax-Friedrichs scheme and report on the stability and efficiency of the scheme by performing numerical experiments. The computed result satisfies some well known qualitative features of the solution.

Keywords: Traffic Flow Model, Nonlinear PDE, Numerical Simulation.

1 Introduction

Traffic management is becoming a mounting challenge for many cities and towns across the globe as the world's population grows. Increasing attention has been devoted to the modeling, simulation, and visualization of traffic flows to investigate causes of traffic congestion and accidents, to study the effectiveness of roadside hardware, signs and other barriers, to improve policies and guidelines related to traffic regulation and to assist urban development and the design of highway and road systems. The transportation infrastructure is one of the major pillars supporting life in cities and regions. In many large cities the potentialities of extensive development of the transportation network were exhausted over the last decades or are approaching completion. That is why optimal planning of the transportation, improvement of traffic organization and optimization of the routes of public conveyances take on special importance. Solution of these problems cannot do without mathematical modeling of the transportation system. Dong [1] developed a continuum traffic flow model pertaining to the modeling of discontinuities of traffic flow at bottlenecks on multi-lane freeway and in urban networks. Recently, authors [2, 3] presented a macroscopic model of mixed multi-lane freeway traffic that can be easily calibrated to empirical traffic data, as is shown for Dutch highway data. The model is derived from a gas-kinetic level of description, including effects of vehicular space requirements and velocity correlations between successive vehicles. The optimal velocity model, as modified by Devis [4], which is used in simulations of traffic on a dual-lane highway and a single-lane highway with an on-ramp. The equilibrium solutions of the modified model cover a two-dimensional region of flow-density space beneath the fundamental-diagram curve, rather than just lying on the curve as in the original model.

Klar et al. [5] presented a hierarchy of multilane traffic flow models and described the derivation of multilane traffic flow model. A numerical simulation of multilane traffic flow with the kinetic model was performed by Klar and Wegner [5].

In order to investigate efficient numerical simulation of a single-lane highway traffic flow model by using finite difference scheme we have studied the works [7], [8], [10]. In this article, we extend the work for a multilane traffic flow model (macroscopic type) appended with a linear density-velocity relationship. The derivation of the macroscopic multilane traffic model is presented (see also [1], [2]). The multilane traffic flow model consists of a system of nonlinear partial differential equations (PDE) and it is almost impossible to find the exact solution of the model as an IVP. That is why there is a demand to get the numerical solution of the multilane model as an IBVP. For numerical solution of the model we describe the derivation of the Lax-Friedrichs scheme for the considered model (see section 3). In Section 4, in order to implement the numerical scheme we develop a computer programming code and perform numerical simulation of the multilane traffic flow with respect to various flow parameters. Finally, we verify the stability condition and a physical constraint condition of the Lax-Friedrichs scheme for the considered model.

2 Macroscopic Multilane Traffic Flow Model

Typically, continuum models for multilane traffic are based on a system of conservation laws with source terms. We assume a highway with N lanes, they are numbered by $\alpha = 1, \cdots, N$. In our research, we focus on the macroscopic multilane traffic model which can be written in generalized form as follows from [5].

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_j}{\partial t} + \frac{\partial q_j}{\partial x} &= \frac{\rho_{j-1}}{T_{j-1}^j} - \frac{\rho_j}{T_j^{j-1}} + \frac{\rho_{j+1}}{T_{j+1}^j} - \frac{\rho_j}{T_j^{j+1}} \\ \frac{\partial \rho_N}{\partial t} + \frac{\partial q_N}{\partial x} &= \frac{\rho_{N-1}}{T_{N-1}^N} - \frac{\rho_N}{T_N^{N-1}} \end{aligned} \qquad (2.1)$$

Here, the subscripts $1, j = 2, \dots, N-1$ and N refer to the lane numbers. The quantities ρ_j and $q_j = \rho_j v_j$ are the vehicle density and the vehicle flux in the j-th lane respectively whereas v_j is the vehicle velocity at the j-th lane for $j = 1, \dots, N$; at last $T_j^k = T_j^k(\rho_j, \rho_k)$ is the vehicle transition rate from lane j to lane k, with $|j-k| = 1$.

The macroscopic multilane traffic flow model (2.1) is approximated by the Greenschield's linear density-velocity relationship as follows:

$$V(\rho) = V_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \qquad (2.2)$$

Where, V_{max} is the maximum velocity and ρ_{max} is the maximum density which is based on bumper to bumper traffic.

3 Numerical Solution of the Model

In particular, we choose macroscopic multilane traffic flow model (2.1) for two lanes that is for $j = 1, 2$ $(N = 2)$. In that case the multilane traffic model (2.1) takes the following form.

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial q_2}{\partial x} &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \end{aligned} \tag{3.1}$$

For numerical solution of the model we have to make the model as an IBVP by inserting initial and boundary conditions. We use Lax-Friedrichs scheme to find the numerical solution. The IBVP is given by

$$\begin{gathered} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} = \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial q_2}{\partial x} = \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \\ \rho_1(0, x) = (\rho_1)_0(x),\ \rho_2(0, x) = (\rho_2)_0(x) \\ \rho_1(t, a) = (\rho_1)_a(t),\ \rho_2(t, a) = (\rho_2)_a(t),\ \rho_1(t, b) = (\rho_1)_b(t), \rho_2(t, b) = (\rho_2)_b(t) \end{gathered} \tag{3.2}$$

where $t_0 \leq t \leq T$ and $a \leq x \leq b$. In order to develop the scheme, we discretize the time derivatives, $\frac{\partial \rho_1}{\partial t}$ and $\frac{\partial \rho_2}{\partial t}$ by the forward difference in time and the space derivatives $\frac{\partial q_1}{\partial x}$ and $\frac{\partial q_2}{\partial x}$ by the central differences in space as in [9].

Forward difference in time: According to the Taylor series, we obtain

$$\left.\begin{aligned} \frac{\partial \rho_1}{\partial t} &\approx \frac{\rho_1(t + k, x) - \rho_1(t, x)}{k} \\ \frac{\partial \rho_2}{\partial t} &\approx \frac{\rho_2(t + k, x) - \rho_2(t, x)}{k} \end{aligned}\right\} \tag{3.3}$$

Central Difference in space:

Here we also apply the Taylor Series and obtain

$$\left.\begin{aligned} \frac{\partial q_1}{\partial x} &\approx \frac{q_1(t, x + h) - q_1(t, x - h)}{2h} \\ \frac{\partial q_2}{\partial x} &\approx \frac{q_2(t, x + h) - q_2(t, x - h)}{2h} \end{aligned}\right\} \tag{3.4}$$

We assume the uniform grid spacing with step size k and h for time and space respectively $t^{n+1} = t^n + k$ and $x_{i+1} = x_i + h$. We write $(\rho_1)_i^n$ for $\rho_1(t, x)$ and $(\rho_2)_i^n$ for $\rho_2(t, x)$ in equations of (3.3). We also write $(q_1)_i^n$ for $q_1(t, x)$ and $(q_2)_i^n$ for

$q_2(t, x)$ in equations of (3.4). Inserting (3.3) and (3.4) in the equations of IBVP (3.2), the discrete versions of the equations are as follows from [9].

$$\frac{{\rho_1}_i^{n+1} - {\rho_1}_i^n}{k} + \frac{{q_1}_{i+1}^n - {q_1}_{i-1}^n}{2h} = \frac{{\rho_2}_i^n}{T_2^1} - \frac{{\rho_1}_i^n}{T_1^2} \tag{3.5}$$

Then we rewrite equation (3.5) as

$$\rho_{1i}^{n+1} = \rho_{1i}^n - \frac{k}{2h}\left({q_1}_{i+1}^n - {q_1}_{i-1}^n\right) + k\left(\frac{{\rho_2}_i^n}{T_2^1} - \frac{{\rho_1}_i^n}{T_1^2}\right) \tag{3.6}$$

Similarly,

$$\rho_{2i}^{n+1} = \rho_{2i}^n - \frac{k}{2h}\left({q_2}_{i+1}^n - {q_2}_{i-1}^n\right) + k\left(\frac{{\rho_1}_i^n}{T_1^2} - \frac{{\rho_2}_i^n}{T_2^1}\right) \tag{3.7}$$

Where

$${q_1}_i^n = V_{max}\left({\rho_1}_i^n - \frac{({\rho_1}_i^n)^2}{\rho_{max}}\right) \text{ and } {q_2}_i^n = V_{max}\left({\rho_2}_i^n - \frac{({\rho_2}_i^n)^2}{\rho_{max}}\right)$$

The equations (3.6) and (3.7) are the Lax-Friedrichs scheme for the IBVP (3.2).

4 Numerical Simulation

In this section, we present the numerical results for some specific cases of traffic flow focusing on the traffic flow parameters by using the Lax-Friedrichs scheme. For a particular case, we choose maximum velocity, ${V_1}^{max} = {V_2}^{max} = 60$ km/hour. It is notified that for satisfying the CFL condition we pick the unit of velocity as km/sec. We consider maximum density, ${\rho_1}^{max} = {\rho_2}^{max} = 180$ /km, Transition rates, $T_2^1 = 20\%$, $T_1^2 = 10\%$ and perform numerical experiment for 6 minutes in $\delta t = 2400$ time steps for a two lanes highway of 10 km in 401 spatial grid points with step sizes $\delta x = 100$ meters. We consider the initial density ${\rho_1}^0$ and ${\rho_2}^0$ as presented in Fig: 4.1. The computed density profile is also presented in Fig: 4.2.

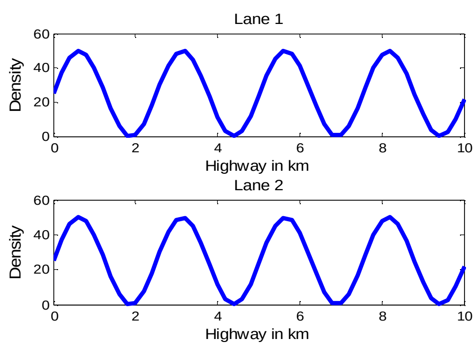

Fig 4.1: Initial Density

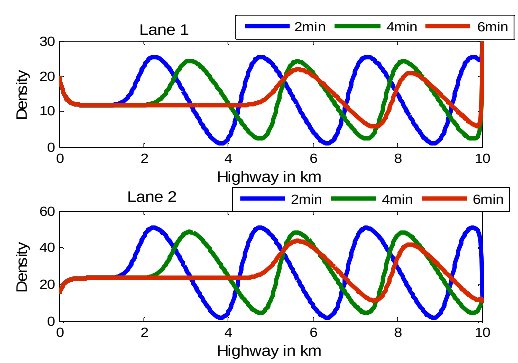

Fig 4.2: Density Profile

Using Lax-Friedrichs scheme we get the density for both the lanes of traffic flow model. We can also compute the velocity from the density with the aid of the chosen density-velocity relationship. The computed velocity is depicted with respect to the distance in the following Fig: 4.3.

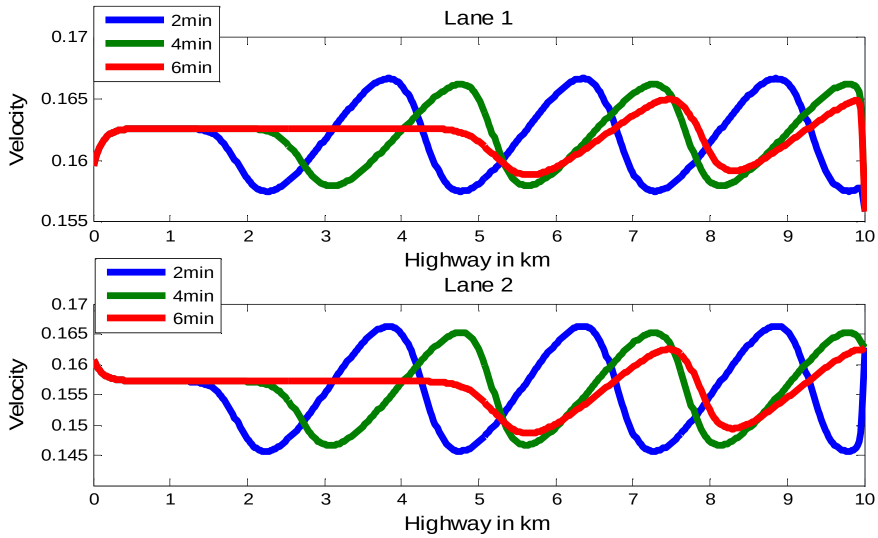

Fig 4.3: Computed Velocity

We reduce the parameter of maximum velocity for both the lanes for which traffic much slower than the previous case and when we increase the parameter of maximum density for both the lanes, the traffic density has no significant change that is almost as previous. In that case, the density profile is presented in Fig 4.4.

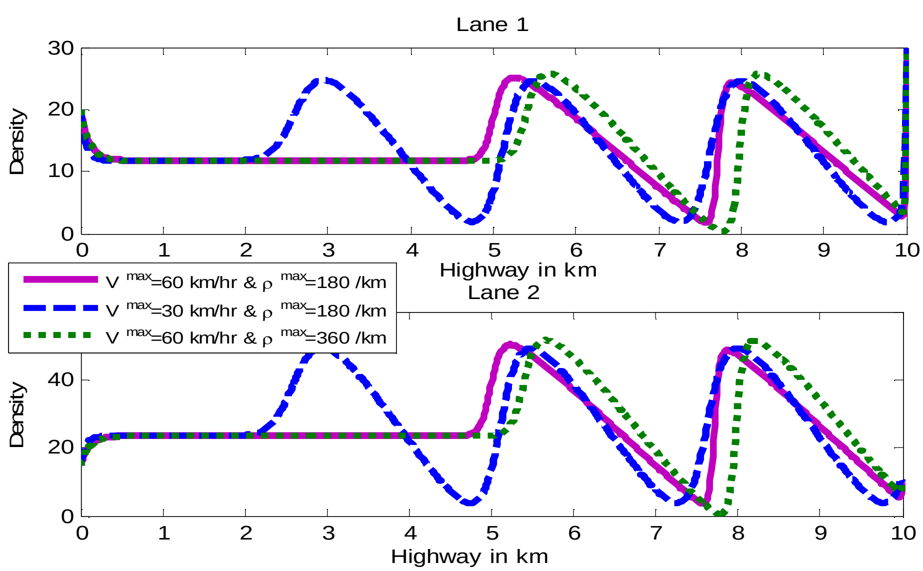

Fig 4.4: Density Profile for larger ρ_{max} and smaller V_{max}

The numerical solution of the multilane traffic flow model for different spatial and temporal grid size is depicted in figure 4.5. It shows that the numerical solution of the model converges with respect to the smaller spatial grid size, δx and temporal grid size, δt which is a very good agreement of the numerical scheme of the multilane traffic model.

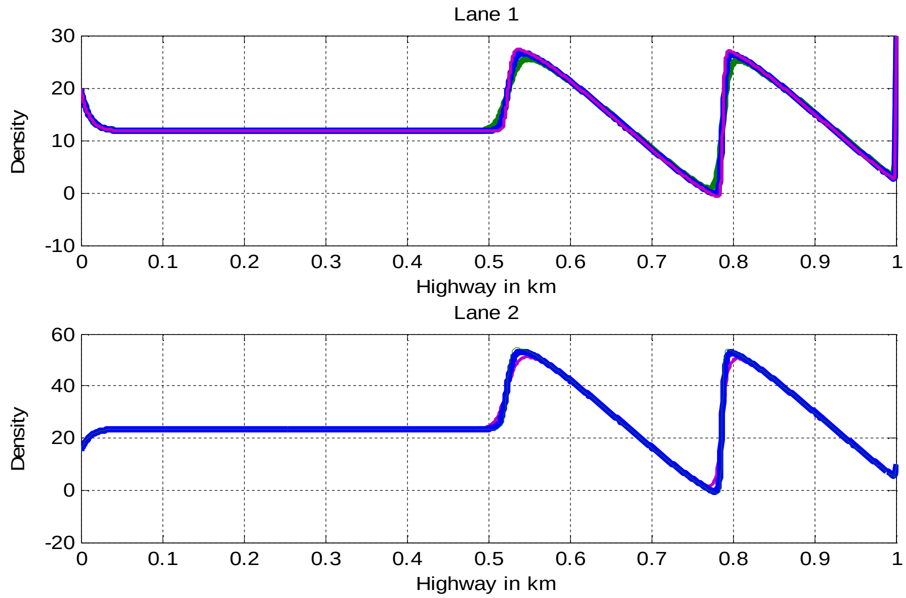

Fig 4.5: Density Profile for different spatial and temporal size

The fundamental diagram of traffic flow is one of the important characteristics of fluid dynamic traffic flow model. Figure 4.6 presents the graphs of Flux for both lanes with respect to the density which are known as the fundamental diagram of traffic flow. The 3D visualization of the density for both the lanes is depicted in the Fig: 4.7.

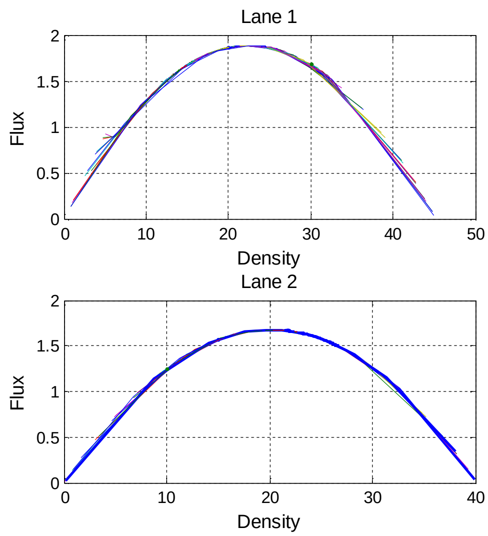

Fig 4.6: Fundamental Diagram of Traffic Flow

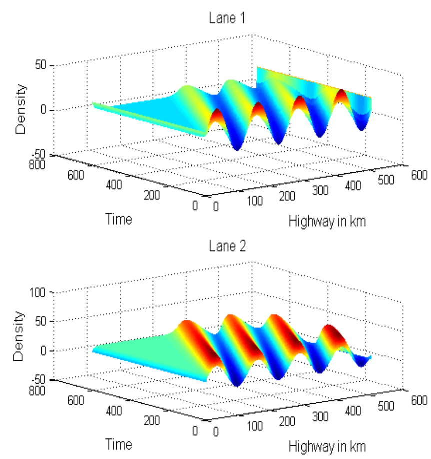

Fig 4.7: Solution Surface

In this context, we investigate stability condition and physical constraint condition for the numerical scheme. The physical constraint and stability condition of Lax-Friedrichs scheme for single lane traffic flow model follows from [8] is

$$V_{max}\frac{\Delta t}{\Delta x} \leq 1 \quad \text{and} \quad \rho_{max} = k \max_{j} \rho_0(x_i); \quad k \geq 2$$

In case of multilane traffic flow model, experimentally we inspect that the stability condition of Lax-Friedrichs scheme also remain unchanged for our considered multilane traffic flow model.

5 Conclusion

We have described the derivation of Lax-Friedrichs scheme for the numerical solution of the of the multilane traffic flow model as an IBVP. We have developed a computer program code for the numerical scheme in order to perform numerical simulation with respect to various traffic flow parameters. The Lax-Friedrichs scheme has provided the fundamental diagram of traffic flow which is a very good qualitative agreement of the analytical theory of the model. Finally, we have inspected experimentally that the physical constraint and the stability condition of the Lax-Friedrichs scheme for the single lane traffic flow is identical as that of our considered model.

REFERENCES

[1] Dong Ngoduy, Macroscopic Discontinuity Modeling for Multiclass Multilane Traffic Flow Operations, Master of Science in Engineering Thesis, Delft University of Technology, Netherland, 2006

[2] V. I. Shvetsov, Mathematical Modeling of Traffic Flows, *Automation and Remote Control*, Vol. 64, No. 11, pp. 1651-1689, 2003

[3] Vladimir Shvetsov and Dirk Helbing, Macroscopic Dynamics of Multi-Lane Traffic, *Physical Review*, E59, pp. 6328-6339, 1999

[4] L. C. Davis, Multilane Simulations Of Traffic Phases, *Physical Review E 69*, 016108, 2004

[5] Axel Klar and Raimund Wegner, A Hierarchy of Models for Multilane Vehicular Traffic I: Modeling, *SIAM Journal on Applied Mathematics*, Vol. 59, No. 3, pp. 983-1001, 2010

[6] Muhammad Humayun Kabir, Numerical Simulation of Multilane Traffic Flow Model**,** M. Phil. Thesis, Department of Mathematics, Jahangirnagar University, 2012

[7] M. H. Kabir, M. O. Gani and L. S. Andallah, Numerical Simulation of Mathematical Model of Traffic flow based on a Nonlinear Density-Velocity Function, *Journal of Bangladesh Academy of Sciences*, Vol. 34, no. 1, pp. 15-22, 2010

[8] Mohammad Osman Gani, A Finite Difference Method for Fluid Dynamic Traffic Flow Model with Two Sided Boundary Conditions**,** M. Phil. Thesis, Department of Mathematics, Jahangirnagar University, 2010

[9] Randall J. LeVeque, Numerical Methods for Conservation Laws, Basel; Boston; Berlin: Birkhauser, 1992, second edition

[10] M H Kabir, A Afroz and L S Andallah, A Finite Dfference Scheme for a Macroscopic Traffic Flow Model with Non-linear Density-Velocity Realtionship, *Bangladesh Journal of Scientific and Industrial Research*, Vol. 47(3), pp. 339-347, 2012