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TRAVELING WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATION VIA ENHANCED (G'/G)-EXPANSION METHOD

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ABSTRACT

In this article, the Enhanced (G'/G)-expansion method has been projected to find the traveling wave solutions for nonlinear evolution equations(NLEEs) via the (2+1)-dimensional Burgers equation. The efficiency of this method for finding these exact solutions has been demonstrated with the help of symbolic computation software Maple. By this method we have obtained many new types of complexiton soliton solutions, such as, various combinations of trigonometric periodic function and rational function solutions, various combination of hyperbolic function and rational function solutions. The proposed method is direct, concise and effective, and can be used for many other nonlinear evolution equations.

Key Words: Enhanced (G'/G)-expansion method, The (2+1)-dimensional Burgers equation, Traveling wave solutions, complexiton soliton, NLEEs.

Mathematics Subject Classification: 35K99, 35P05, 35P99.

1. Introduction

Nowadays NLEEs have been the subject of all-embracing studies in various branches of nonlinear sciences. A special class of analytical solutions named traveling wave solutions for NLEEs has a lot of importance, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are frequently used to describe many problems of protein chemistry, chemically reactive materials, in ecology most population models, in physics the heat flow and the wave propagation phenomena, quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibers, biology, solid state physics, chemical kinematics, geochemistry, meteorology, electricity etc. Therefore investigating traveling wave solutions is becoming more and more attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as the Hirota's bilinear transformation method [1, 2], the tanh-function method [3, 4], the extended tanh-method [5, 6], the Exp-function method [7-12], the Adomian decomposition method [13], the F-expansion method [14], the auxiliary equation method [15], the Jacobi elliptic function method [16], the modified Exp-function method [17], the (G'/G)-expansion method [18-23], the Modified simple equation method [24-26] and so on.

Various ansatz have been proposed for seeking traveling wave solutions of nonlinear differential equations.

He et al. [7] have introduced the Exp-function method which is very simple and straightforward. It is based on the assumption that traveling wave solutions can be expressed in the following form [7-12]:

$$u(\xi)=\frac{\sum_{n=-c}^{d}A_n\exp(n\xi)}{\sum_{m=-p}^{q}B_m\exp(m\xi)}=\frac{A_{-c}\exp(-c\xi)+\cdots\cdots+A_d\exp(d\xi)}{B_{-p}\exp(-p\xi)+\cdots\cdots+B_q\exp(q\xi)},$$

where c, d, p and q are positive integers which are unknown to be determined, A_n and B_m are unknown constants. They had applied the Exp-function method for exact traveling wave solutions of modified KdV equation and Dodd–Bullough–Mikhailov equation.

Recently, Wang et al. [20] have introduced a simple method which is called the (G'/G)-expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are arbitrary constants and $u(\xi) = \alpha_m\left(\frac{G'}{G}\right)^m + \cdots\cdots\cdots$ be the traveling wave solution of NLEEs. By means of this method they have solved the KdV equation, the mKdV equation, the variant Boussinesq equations and the Hirota–Satsuma equations.

Guo et al. [23] have introduced an another method so called extended (G'/G)-expansion method where $G=G(\xi)$ satisfies the second order linear ordinary differential equation $G''+\mu G=0$, where $G'=\frac{dG(\xi)}{d\xi}$, $G''=\frac{d^2G(\xi)}{d\xi^2}$, $\xi = x - Vt$, V is a constant. They choose the traveling wave solution of the form

$$u(\xi)=a_0+\sum_{i=1}^{n}\left(a_i(G'/G)^i+b_i(G'/G)^{i-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}\right).$$

They have proposed extended (G'/G)-expansion method to construct traveling wave solutions of Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations.

For further references for the (G'/G)- expansion method see the articles [18-23].

In direct methods, the choice of an appropriate ansatz is of great importance. In this paper, based on a new general ansatz, we propose the Enhanced (G'/G)-expansion method which can be used to obtain explicit solutions of NLEEs.

Among those approaches, an Enhanced (G'/G)-expansion method is a tool to reveal the solitons and periodic wave solutions of NLEEs in mathematical physics and engineering. The main ideas of the Enhanced (G'/G)-expansion method are that the traveling wave solutions of NLEEs can be expressed as rational functions of (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \mu G = 0$.

The objective of this article is to apply the Enhanced (G'/G)-Expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the (2+1)-dimensional Burgers equations.

The article is prepared as follows: In section 2, the Enhanced (G'/G)-Expansion method is discussed. In section 3, we apply the Enhanced (G'/G)-Expansion method to the nonlinear evolution equations pointed out above; in section 4, physical explanations and in section 5 conclusions are given.

2. The Enhanced (G'/G)-Expansion method

In this section we describe the Enhanced (G'/G)-Expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$\mathcal{R}(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots\dots\dots) = 0, \tag{2.1}$$

where $u(\xi) = u(x,t)$ is an unknown function, $\mathcal{R}$ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1. Combining the independent variables x and t into one variable $\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x,t), \quad \xi = x \pm \omega t . \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ODE:

$$\wp(u, u', u'', \dots\dots\dots) = 0, \tag{2.3}$$

where $\wp$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. We suppose that Eq.(2.3) has the formal solution

$$u(\xi) = \sum_{i=-n}^{n} \left(\frac{a_i (G'/G)^i}{(1+\lambda(G'/G))^i} + b_i (G'/G)^{i-1} \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu}\right)} \right), \tag{2.4}$$

Where $G = G(\xi)$ satisfy the equation $G'' + \mu G = 0$, (2.5) in which $a_i, b_i\ (-n \le i \le n; n \in \mathbb{N})$ and λ are constants to be determined later, and $\sigma = \pm 1, \mu \neq 0$.

Step 3. We determine the positive integer n in Eq. (2.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (2.3).

Step 4. We substitute Eq. (2.4) into Eq. (2.3) using Eq. (2.5) and then collect all terms of same powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}$ together, then set each coefficient of them to zero to yield a over-determined system of algebraic equations and then solve this system for $a_i, b_i(-n \le i \le n; n \in \mathbb{N})$ and λ, ω.

Step 5. For the general solutions of Eq. (2.5), we can get

$$\frac{G'}{G} = \sqrt{-\mu}\left(\frac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) = \mathrm{H}_1;\ \text{when}\ \mu < 0.$$

and
$$\frac{G'}{G} = \sqrt{\mu}\left(\frac{A\sin(\sqrt{\mu}\xi)-B\cos(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) = \mathrm{H}_2;\ \text{when}\ \mu > 0. \tag{2.6}$$

where A, B are arbitrary constants. Finally, substitute $a_i, b_i(-n \le i \le n; n \in \mathbb{N})$, λ, ω and Eq. (2.6) into Eq. (2.4) and obtain traveling wave solutions of Eq. (2.1).

3. Application

In this section, we will exert Enhanced (G'/G)-expansion method to solve the (2+1)-dimensional Burgers equation in the form,

$$u_t - uu_x - u_{xx} - u_{yy} = 0. \tag{3.1}$$

The traveling wave transformation equation $u(\xi) = u(x,t), \xi = x - \omega t$ reduces Eq.(3.1) to the following ordinary differential equation:

$$-\omega u' - uu_x - u_{xx} - u_{yy} = 0. \tag{3.2}$$

Integrating Eq.(3.2) with respect to ξ, we get

$$\omega u + \frac{u^2}{2} + 2u' + k = 0, \tag{3.3}$$

where k is an integration constant.

Now balancing the highest-order derivative u' and nonlinear term u^2 in Eq. (3.3), we get $2n = n + 1$, which gives $n = 1$. Therefore, the solution Eq. (2.4) reduces to

$$u(\xi) = a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}[1+\lambda(G'/G)]}{(G'/G)} + b_0(G'/G)^{-1}\sqrt{\sigma\left[1+\frac{(G'/G)^2}{\mu}\right]}$$
$$+b_1\sqrt{\sigma\left[1+\frac{(G'/G)^2}{\mu}\right]} + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left[1+\frac{(G'/G)^2}{\mu}\right]}, \tag{3.4}$$

where $G = G(\xi)$ satisfies Eq. (2.5). Substituting Eq. (3.4) along with Eq. (2.5) into Eq. (3.3), collecting all terms with the same powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left[1+\frac{(G'/G)^2}{\mu}\right]}$

and setting them to zero, we get over-determined system of eighteen algebraic equations with the aid of software Maple.

Set 1: $k = \frac{1}{2}a_0^2 + 4a_0\lambda\mu + 8\mu^2\lambda^2 + 8\mu, \omega = -4\mu\lambda - a_0, \lambda = \lambda, a_0 = a_0,$
$a_1 = 4\mu\lambda^2 + 4, a_{-1} = b_{-1} = b_0 = b_1 = 0.$

Set 2: $k = \frac{1}{2}a_0^2 - 4a_0\lambda\mu + 8\mu^2\lambda^2 + 8\mu, \omega = 4\mu\lambda - a_0, \lambda = \lambda, a_{-1} = -4\mu,$
$a_0 = a_0, a_1 = b_{-1} = b_0 = b_1 = 0.$

Set 3: $k = \frac{1}{2}a_0^2 + 32\mu, \omega = -a_0, \lambda = 0, a_{-1} = -4\mu, a_0 = a_0, a_1 = 4, b_{-1} = b_0 =$ $b_1 = 0.$

Set 4: $k = \frac{1}{2}a_0^2 - 2a_0\lambda\mu + 2\mu^2\lambda^2 + 2\mu, \omega = 2\mu\lambda - a_0, \lambda = \lambda, a_{-1} = -2\mu,$
$a_0 = a_0, b_0 = \pm\frac{2}{\sqrt{\sigma}}, a_1 = b_{-1} = b_1 = 0.$

Now substituting Set 1-Set 4 and Eq. (2.6) into Eq. (3.4), we deduce abundant traveling wave solutions of Eq. (3.1) as follows:

For another set $\{k = -\frac{1}{2}a_0(a_0 + 2\omega), \omega = \omega, \lambda = \lambda, a_0 = a_0, a_{-1} = a_1 = b_{-1} = b_0 = b_1 = 0\}$ Eq.(3.4) gives trivial solutions.So this case is rejected.

When $\mu < 0$, $A \neq B$, we have the following hyperbolic function solutions:

Family- 1: $u(\xi) = a_0 + \frac{(4\mu\lambda^2+4)\mathrm{H}_1}{1+\lambda\mathrm{H}_1},$

where $\xi = x + y - (-4\mu\lambda - a_0)t.$

Family- 2: $u(\xi) = a_0 - \frac{4\mu(1+\lambda\mathrm{H}_1)}{\mathrm{H}_1},$

where $\xi = x + y - (4\mu\lambda - a_0)t.$

Family- 3: $u(\xi) = a_0 + \frac{4\mathrm{H}_1}{1+\lambda\mathrm{H}_1} - \frac{4\mu(1+\lambda\mathrm{H}_1)}{\mathrm{H}_1},$

where $\xi = x + y - (-a_0)t.$

Family- 4: $u(\xi) = a_0 - \frac{2\mu(1+\lambda\mathrm{H}_1)}{\mathrm{H}_1} \pm \frac{2}{\sqrt{\sigma}}(\mathrm{H}_1)^{-1}\sqrt{\sigma\left(1 + \frac{\mathrm{H}_1^2}{\mu}\right)},$

where $\xi = x + y - (2\mu\lambda - a_0)t.$

when $\mu > 0$, we have the following trigonometric function solutions for the above mentioned set 1 to set 4.

Family- 5: $u(\xi) = a_0 + \frac{(4\mu\lambda^2+4)\mathrm{H}_2}{1+\lambda\mathrm{H}_2},$

where $\xi = x + y - (-4\mu\lambda - a_0)t.$

Family- 6: $u(\xi) = a_0 - \frac{4\mu(1+\lambda\mathrm{H}_2)}{\mathrm{H}_2},$

where $\xi = x + y - (4\mu\lambda - a_0)t.$

Family- 7: $u(\xi) = a_0 + \frac{4H_2}{1+\lambda\mathrm{H}_2} - \frac{4\mu(1+\lambda\mathrm{H}_2)}{\mathrm{H}_2},$

where $\xi = x + y - (-a_0)t$.

Family- 8: $$u(\xi) = a_0 - \frac{2\mu(1+\lambda H_2)}{H_2} \pm \frac{2}{\sqrt{\sigma}}(H_2)^{-1}\sqrt{\sigma\left(1 + \frac{H_2^2}{\mu}\right)},$$

where $\xi = x + y - (2\mu\lambda - a_0)t$.

4. Physical Explanations

In this section we will discuss the physical explanations and graphical representation of the above determined eight families of solutions.

4.1 Explanations

The introduction of dispersion without introducing nonlinearity destroys the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also prevents the formation of solitary waves, because the pulse energy is frequently pumped into higher frequency modes. However, if both dispersion and nonlinearity are present, solitary waves can be sustained. Similarly to dispersion, dissipation can also give rise to solitary waves when combined with nonlinearity. Hence it is interesting to point out that the delicate balance between the nonlinearity effect of uu_x and the dissipative effect of u_{xx} and u_{yy} give rise to solitons, that after a fully interaction with others the solitons come back retaining their identities with the same speed and shape. The (2+1)-dimensional Burgers equation has solitary wave solutions that have exponentially decaying wings. If two solitons of the (2+1)-dimensional Burgers equation collide, the solitons just pass through each other and emerge unchanged. For special values of the parameters solitary wave solutions are originated from the obtained exact solutions.

Fig. 1: For the values of $\mu = -1, \lambda = 0, A = 1, B = 0, a_0 = 1, y = 0$ within the interval $-3 \le x, t \le 3$, Family-1 represents kink wave. Kink waves are traveling waves which rise or descend from one asymptotic state to another. The kink solution approaches a constant at infinity.

Fig. 2: For the values of $\mu = -2, \lambda = 1, A = 2, B = 1, a_0 = 1, y = 0$ within the interval $-1 \le x, t \le 1$, Family-2 represents soliton wave. Solitons are special kinds of solitary waves. The soliton solution is spatially localized solution, hence $u'(\xi)$, $u''(\xi)$ and $u'''(\xi) \to 0$ as $\xi \to \pm\infty$, $\xi = x - \omega t$. Solitons have a remarkable soliton property in that it keeps its identity upon interacting with other solitons.

Fig. 3: For the values of $\mu = -1, A = 0, B = 1, a_0 = 1, y = 0$ within the interval $-10 \le x, t \le 10$, Family-3 represents singular soliton wave.

Fig. 4: For the values of $\mu = -1, \lambda = 2, A = 0, B = 1, a_0 = 0, y = 0$ within the interval $-10 \le x, t \le 10$, Family-4 represents complex kink wave under the condition that any one of the constant A or B must be zero.

Family-5 to Family-8 are periodic solutions. Periodic solutions are traveling wave solutions that are periodic.

Fig. 5: For the values of $\mu = 1, \lambda = 0, A = 1, B = 1, a_0 = 1, y = 0$ within the interval $-3 \leq x, t \leq 3$ Fig. 5 represents the shape of Family-5.

Fig. 6: For the values of $\mu = 2, \lambda = 1, A = 1, B = 0, a_0 = 1, y = 2$ within the interval $-10 \leq x, t \leq 10$ Fig. 6 represents the shape of Family-6.

Fig. 7: For the values of $\mu = 1, \lambda = 0, A = 0, B = 1, a_0 = 2, y = 2$ within the interval $-5 \leq x, t \leq 5$ Fig. 7 represents the shape of Family-7.

Fig. 8: For the values of $\mu = 1, \lambda = 1, A = 1, B = 3, a_0 = 0, y = 0$ within the interval $-3 \leq x, t \leq 3$ Fig. 8 represents the shape of Family-8.

Moreover, the graphical illustrations of some obtained solutions are shown in Fig.-1 to Fig.-8 in the following subsection.

4.2. Graphical representation

Some of our obtained traveling wave solutions are represented in the following figures with the aid of commercial software Maple:

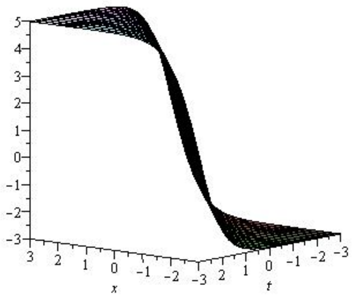

Fig.1: Kink solution of Family-1 for $\mu = -1, \lambda = 0, A = 1, B = 0, a_0 = 1, y = 0, -3 \leq x, t \leq 3$.

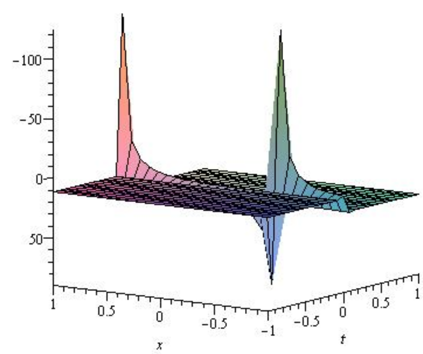

Fig.2: Soliton solution of Family-2 for $\mu = -2, \lambda = 1, A = 2, B = 1, a_0 = 1, y = 0, -1 \leq x, t \leq 1$.

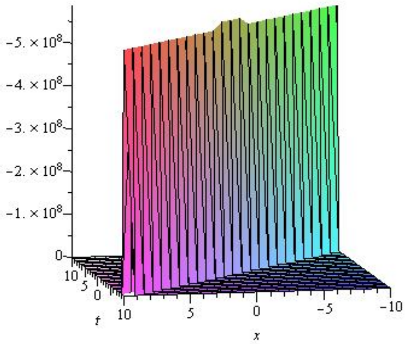

Fig.3: Soliton solution of Family-3 for $\mu = -1, A = 0, B = 1, a_0 = 1, y = 0, -10 \leq x, t \leq 10$.

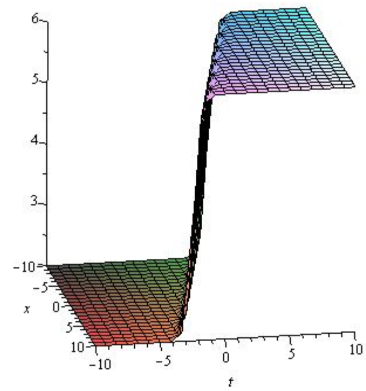

Fig.4: Kink solution of Family-4 for $\mu = -1, \lambda = 2, A = 0, B = 1, a_0 = 0, y = 0, -10 \leq x, t \leq 10$.

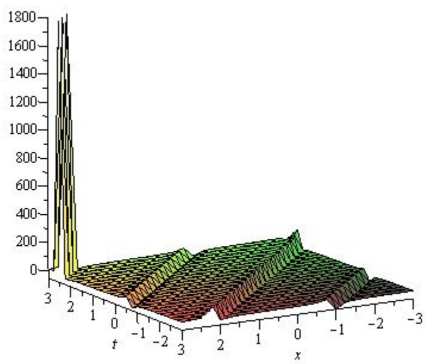

Fig.5: Periodic solution of Family-5 for $\mu = 1, \lambda = 0, A = 1, B = 1, a_0 = 1, y = 0, -3 \leq x, t \leq 3$.

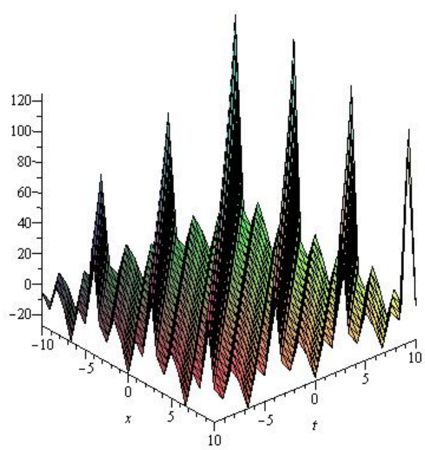

Fig.6: Periodic solution of Family-6 for $\mu = 2, \lambda = 1, A = 1, B = 0, a_0 = 1, y = 2, -10 \leq x, t \leq 10$.

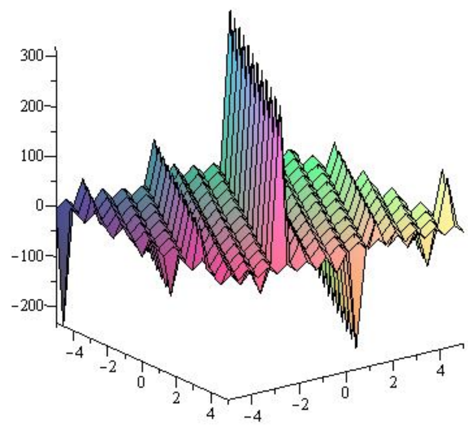

Fig.7: Periodic solution of Family-7 for $\mu = 1, \lambda = 0, A = 0, B = 1, a_0 = 2, y = 2, -5 \leq x, t \leq 5$.

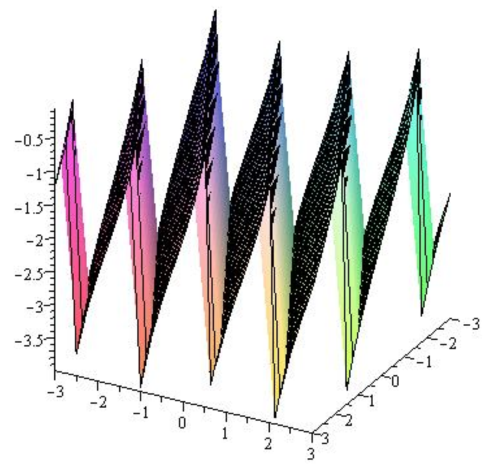

Fig.8: Periodic solution of Family-8 for $\mu = 1, \lambda = 1, A = 1, B = 3, a_0 = 0, y = 0, -3 \leq x, t \leq 3$.

5. Conclusion

In summary, we have proposed the Enhanced (G'/G)-expansion method and applied it to the (2+1)-dimensional Burgers equation. As a result, some new exact traveling wave solutions, so called complexiton soliton solutions are obtained. The method which we have proposed in this letter is standard, direct and computerized method which allows us to do complicated and tedious algebraic calculation. It is shown that the algorithm used in this paper can be also applied to other NLEEs in mathematical physics.

REFERENCES

[1]. Hirota R., *Exact envelope soliton solutions of a nonlinear wave equation*. J. Math. Phy. 14(1973) 805-810.

[2]. Hirota R., Satsuma J., *Soliton solutions of a coupled KDV equation*. Phy. Lett. A. 85(1981) 404-408.

[3]. Malfliet M., *Solitary wave solutions of nonlinear wave equations*. Am. J. Phys. 60, (1992)650–654.

[4]. Nassar H. A., Abdel-Razek M. A., Seddeek A. K., *Expanding the tanh-function method for solving nonlinear equations*, Appl. Math. 2(2011) 1096-1104.

[5]. Fan E.G., *Extended tanh-method and its applications to nonlinear equations.*Phy. Lett. A. 277(2000) 212-218.

[6]. Abdou M. A., *The extended tanh-method and its applications for solving nonlinear physical models.* App. Math. Comput. 190(2007) 988-996.

[7]. He J.H., Wu X.H., *Exp-function method for nonlinear wave equations,* Chaos, Solitons and Fract. 30(2006) 700-708.

[8]. Akbar M.A., Ali N.H.M., *Exp-function method for Duffing Equation and new solutions of (2+1) dimensional dispersive long wave equations.* Prog. Appl. Math. 1(2)(2011) 30-42.

[9]. Bekir A., Boz A., *Exact solutions for nonlinear evolution equations using Exp-function method*. Phy.Lett. A. 372(2008) 1619-1625.

[10]. Abdou M. A., Soliman A. A. and Basyony S. T., *New application of exp-function method for improved Boussinesq equation*. Phys. Lett.A, 369(2007),469-475.

[11]. El-WakilS. A., Madkour M. A. and Abdou M. A., *Application of exp-function method for nonlinear evolution equations with variable co-efficient*, Phys. Lett. A, 369(2007),62-69.

[12]. Mohyud-Din S. T., Noor M. A. and Waheed A., *Exp-function method for generalized travelling solutions of Calogero-Degasperis-Fokas equation*, Zeitschriftfür Naturforschung A- A Journal of Physical Sciences, 65a (2010), 78-84.

[13]. Adomian G., *Solving frontier problems of physics: The decomposition method*. Boston(1994), M A: Kluwer Academic.

[14]. Zhou Y.B., Wang M.L., Wang Y.M., *Periodic wave solutions to coupled KdV equations with variable coefficients*, Phys. Lett. A. 308(2003) 31–36.

[15]. Sirendaoreji, *New exact travelling wave solutions for the Kawahara and modified Kawahara equations*. Chaos Solitons Fract. 19(2004) 147-150.

[16]. Ali A.T., *New generalized Jacobi elliptic function rational expansion method*. J. Comput. Appl. Math. 235(2011) 4117-4127.

[17]. He Y., Li S., Long Y., *Exact solutions of the Klein-Gordon equation by modified Exp-function method*. Int. Math. Forum. 7(4) (2012) 175-182.

[18]. Akbar M.A., Ali N.H.M., ZayedE. M. E., *Abundant exact traveling wave solutions of the generalized Bretherton equation via* (G'/G)*--expansion method*. Commun. Theor. Phys. 57(2012a) 173-178.

[19]. Akbar M.A., Ali N.H.M., Zayed E. M. E., *A generalized and improved* (G'/G)*—expansion method for nonlinear evolution equations*, Math. Prob. Engr., Vol. 2012, 22 pages. doi: 10.1155/2012b/459879.

[20]. Wang M., Li X., Zhang J., *The* (G'/G)*-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*. Phys. Lett. A. 372(2008) 417-423.

[21]. Akbar M.A., Ali N.H.M, *The alternative* (G'/G)*-expansion method and its applications to nonlinear partial differential equations*. Int. J. Phys. Sci. 6(35)(2011) 7910-7920.

[22]. Shehata A. R., *The traveling wave solutions of the perturbed nonlinear Schrodinger equation and the cubic-quinticGinzburg Landau equation using the modified* (G'/G)*-expansion method*. Appl. Math. Comput. 217(2010), 1-10.

[23]. Guo S., Zhou Y., *The extended* (G'/G)*-expansion method and its applications to the Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations*. Applied Mathematics and Computation 215 (2010) 3214–3221.

[24]. Jawad A. J. M., Petkovic M. D., Biswas A., *Modified simple equation method for nonlinear evolution equations*. Appl. Math. Comput. 217(2010), 869-877.

[25]. Khan K., Akbar M. A., Ali N. H. M. *The Modified Simple Equation Method for Exact and Solitary Wave Solutions of Nonlinear Evolution Equation: The GZK-BBM Equation and Right-Handed Noncommutative Burgers Equations*, ISRN Mathematical Physics, Hindawi Publishing Corporation, Volume 2013, doi: 10.1155/2013/146704, 5 pages.

[26]. Khan K. And Akbar M. A,*Exact and solitary wave solutions for the Tzitzeica-Dodd-Bulloughand the modified KdV-Zakharov-Kuznetsovequations using the modified simple equation method*, Ain Shams Engineering Journal, doi:10.1016/j.asej.2013.01.010.