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THE *k*-DERIVATION ACTING AS A *k*-ENDOMORPHISM AND AS AN ANTI-*k*-ENDOMORPHISM ON SEMIPRIME NOBUSAWA GAMMA RING

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ABSTRACT

This article defines *k*-endomorphism and anti-*k*-endomorphism on Γ_N-rings, and uses the concept of *k*-derivation of Γ_N-rings. Considering M as a semiprime Γ_N-ring and d as a *k*-derivation of M, it aims to prove that (i) if d acts as a *k*-endomorphism on M such that $M\Gamma M=M$ and $xk(\alpha)x=0$ for all $x\in M$ and $\alpha\in\Gamma$, then $d=0$; and (ii) if d is acting as an anti-*k*-endomorphism on M such that $M\Gamma M=M$, $xk(\alpha)x=0$ and $k(\alpha)x\alpha=\alpha xk(\alpha)$ for all $x\in M$ and $\alpha\in\Gamma$, then $d=0$.

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1. Introduction

First, we excerpt the definition of a Γ-ring introduced by W. E. Barnes [3] which is the generalized form of the original definition given by N. Nobusawa [7].

Definition 1.1 *Let M and* Γ *be additive abelian groups. If there exists a mapping* $(a,\alpha,b)\to a\alpha b$ *of* $M\times\Gamma\times M\to M$ *such that*

(*a*) $(a+b)\alpha c=a\alpha c+b\alpha c$, $a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$,

and (*b*) $(a\alpha b)\beta c=a\alpha(b\beta c)$

are satisfied for all $a,b,c\in M$ *and* $\alpha,\beta\in\Gamma$, *then M is called a* Γ*-ring.*

From definition it is obvious that every ring is a Γ-ring, but the converse is in general not true. For instance, we have

Example 1.1 *Suppose R is a ring with identity 1 and* $M_{m,n}(R)$ *is the set of all* $m\times n$ *matrices over R. Then M is a* Γ*-ring under the usual addition and multiplication of matrices if we choose* $M=M_{m,n}(R)$ *and* $\Gamma=M_{n,m}(R)$.

Now we quote below the introductory definition of gamma ring given by its inventor N. Nobusawa [7] that has been producing an innovative new dimension to generalize the theory of classical rings remarkably.

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Definition 1.2 *Let M be a* Γ*-ring. Additionally, if there exists another mapping* $(\alpha, a, \beta) \rightarrow \alpha a\beta$ *of* $\Gamma \times M \times \Gamma \rightarrow \Gamma$ *such that*

(a^*) $(\alpha + \beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a + b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta + \gamma) = \alpha a\beta + \alpha a\gamma$,

(b^*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$, *and*

(c^*) $a\alpha b = 0$ *implies* $\alpha = 0$

hold for all $a, b, c \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$, *then M is called a* Γ_N*-ring.*

Example 1.2 *Let* $D_{m,n}$ *be the set of all rectangular* $m \times n$ *matrices over a division ring D. If we consider* $M = D_{m,n}$ *and* $\Gamma = D_{n,m}$, *then M is a* Γ_N*-ring under the usual addition and multiplication of matrices.*

Evidently, since the Nobusawa condition (c*) does not hold in a Γ-ring necessarily, we get the following result.

Remark 1.1 *M is a Γ-ring does not imply in general that Γ is an M-ring, but M is a* Γ_N*-ring forces Γ to be an M-ring.*

Considering *M* as a Γ-ring, we recall some useful fundamental preliminary definitions in gamma ring theory as follows.

(i) An additive subgroup *U* of *M* is called a left (or, right) ideal of *M* if and only if $M\Gamma U \subset U$ (or, $U\Gamma M \subset U$), whereas *U* is called a (two-sided) *ideal* of *M* if and only if *U* is a left as well as a right ideal of *M*. **(ii)** *M* is said to be *commutative* if and only if $x\gamma y = y\gamma x$ holds for all $x, y \in M$ *and* $\gamma \in \Gamma$. **(iii)** *M* is called *semiprime* if and only if $a\Gamma M\Gamma a = 0$ implies $a = 0$ for all $a \in M$. **(iv)** The set $C_\alpha = \{c \in M : c\alpha m = m\alpha c$ for all $m \in M\}$ is said to be the α-*center* of *M*, where $\alpha \in \Gamma$ is an arbitrary but fixed element. **(v)** The set $C_\Gamma = \{c \in M : c\alpha m = m\alpha c$ for all $\alpha \in \Gamma$ and $m \in M\}$ is called the *center* of *M*, whence it follows that *M* is commutative if and only if $C_\Gamma = M$. **(vi)** If $a, b \in M$ *and* $\alpha \in \Gamma$, then $[a,b]_\alpha$ is called the *commutator* of *a* and *b* with respect to α, which is defined as $[a,b]_\alpha = a\alpha b - b\alpha a$ (whence it also follows that *M* is commutative if and only if $[a,b]_\alpha = 0$ for all $a,b \in M$ and $\alpha \in \Gamma$).

The following is the definition of *k*-derivation of Γ_N-rings introduced by H. Kandamar in [6] that plays a pivotal role in this article.

Definition 1.3 *Let M be a* Γ_N*-ring, and let* $d : M \rightarrow M$ *and* $k : \Gamma \rightarrow \Gamma$ *be additive mappings. Then d is called a k-derivation of M if and only if* $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ *holds for all* $a, b \in M$ *and* $\alpha \in \Gamma$.

Note that the notions of *k*-isomorphism and anti-*k*-isomorphism of Γ_N-rings are explained in our paper [5]. Based on the nature of endomorphism of rings, we now develop the concepts of *k*-endomorphism and anti-*k*-endomorphism of Γ_N-rings significantly in the following way.

Definition 1.4 *Let M and N be* Γ_N *-rings, and let* $\varphi : M \to N$ *and* $k : \Gamma \to \Gamma$ *be additive surjective mappings. Then* φ *is called* ***(i)*** *a k-homomorphism of M onto N if and only if* $\varphi(a\alpha b) = \varphi(a)k(\alpha)\varphi(b)$ *is satisfied for all* $a, b \in M$ *and* $\alpha \in \Gamma$*; and* ***(ii)*** *an anti-k-homomorphism of M onto N if and only if* $\varphi(a\alpha b) = \varphi(b)k(\alpha)\varphi(a)$ *holds for all* $a, b \in M$ *and* $\alpha \in \Gamma$.

Definition 1.5 *Suppose M and N are* Γ_N *-rings. Then* ***(i)*** *a k-homomorphism* $\varphi : M \to N$ *is called a k-endomorphism on M if and only if* $N = M$*; and* ***(ii)*** *an anti-k-homomorphism* $\varphi : M \to N$ *is called an anti-k-endomorphism on M if and only if* $N = M$.

To be more specific, we conclude that

Remark 1.2 *A k-endomorphism (respectively, an anti-k-endomorphism) on a* Γ_N *-ring M is a k-homomorphism (respectively, an anti-k-homomorphism) of M onto itself.*

In classical ring theory, H. E. Bell and L. C. Kappe [4] proved that if *d* is a derivation of a semiprime ring *R* which is either an endomorphism or an anti-endomorphism on *R*, then $d = 0$; whereas, the behavior of *d* is somewhat restricted in case of prime rings in the way that if *d* is a derivation of a prime ring *R* acting as a homomorphism or an anti-homomorphism on a nonzero right ideal *U* of *R*, then $d = 0$ on *R*.

Afterwards, M. Ş. Yenigül and N. Argaç¸ [9] generalized these results with α-derivations and M. Ashraf et. al. [2] obtained the similar results with (σ, τ)-derivations. Analogously, N. Rehman [8] extended the result for generalized derivations acting on nonzero ideals in case of prime rings. Recently, A. Ali and D. Kumar [1] established the aforementioned result for generalized (θ, φ)-derivations in prime rings.

Here, we extend the above mentioned results following [1, 2, 4, 8, 9] in classical ring theory to those in gamma ring theory with *k*-derivation acting as a *k*-endomorphism or an anti-*k*-endomorphism on semiprime Γ_N -rings. Our objective is to prove that (i) if *d* is a *k*-derivation of a semiprime Γ_N -ring *M* which acts as a *k*-endomorphism on *M* such that $M\Gamma M = M$ and $xk(\alpha)x = 0$ hold for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$; and (ii) if *d* is a *k*-derivation of a semiprime Γ_N -ring *M* acting as an anti-*k*-endomorphism on *M* such that $M\Gamma M = M$, $xk(\alpha)x = 0$ and $k(\alpha)x\alpha = \alpha x k(\alpha)$ hold for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$. In doing so, we go forward as follows.

2. *k*-derivation acting as a *k*-endomorphism

Definition 2.1 *Let M be a* Γ_N *-ring and* $k : \Gamma \to \Gamma$ *an additive surjective mapping. Then a k-derivation* $d : M \to M$ *is said to act as a k-homomorphism on M (meaning that d is acting as a k-homomorphism of M onto itself) if and only if it satisfies* $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b) = d(a)k(\alpha)d(b)$ *holds for all* $a, b \in M$ *and* $\alpha \in \Gamma$.

Lemma 2.1 *Let U be a subring of a* Γ_N *-ring M, and let d be a k-derivation of M acting as a k-homomorphism on U such that* $xk(\alpha)x = 0$ *for every* $x \in U$ *and* $\alpha \in \Gamma$*. Then, for all* $x, y \in U$ *and* $\alpha, \beta \in \Gamma$*:*

(a) $d(x)\beta(x\alpha y - xk(\alpha)d(y)) = 0$ *;(b)* $(x\alpha y - d(x)k(\alpha)y)\beta d(y) = 0$.

Proof. (a) Since *d* acts as a *k*-homomorphism on *U*, for all $x, y \in U$ *and* $\alpha, \beta \in \Gamma$, we have

$$d(x\alpha y) = d(x)\alpha y + xk(\alpha)y + x\alpha d(y) = d(x)k(\alpha)d(y) \quad (1)$$

Putting $x\beta x$ for *x* in (1), we get

$$d(x\beta x)\alpha y + x\beta xk(\alpha)y + x\beta x\alpha d(y) = d(x\beta x)k(\alpha)d(y);$$

$$\Rightarrow d(x)\beta x\alpha y + xk(\beta)x\alpha y + x\beta d(x)\alpha y + x\beta xk(\alpha)y + x\beta x\alpha d(y)$$

$$= d(x)\beta xk(\alpha)d(y) + xk(\beta)xk(\alpha)d(y) + x\beta d(x)k(\alpha)d(y);$$

$$\Rightarrow d(x)\beta x\alpha y + x\beta(d(x)\alpha y + xk(\alpha)y + x\alpha d(y))$$

$$= d(x)\beta xk(\alpha)d(y) + x\beta(d(x)k(\alpha)d(y));$$

$$\Rightarrow d(x)\beta x\alpha y + x\beta d(x\alpha y) = d(x)\beta xk(\alpha)d(y) + x\beta d(x\alpha y);$$

$$\Rightarrow d(x)\beta x\alpha y = d(x)\beta xk(\alpha)d(y);$$

$$\Rightarrow d(x)\beta(x\alpha y - xk(\alpha)d(y)) = 0.$$

(b) Replace *y* by $y\beta y$ in (1) to get

$$d(x)\alpha y\beta y + xk(\alpha)y\beta y + x\alpha d(y\beta y) = d(x)k(\alpha)d(y\beta y)$$

$$\Rightarrow d(x)\alpha y\beta y + xk(\alpha)y\beta y + x\alpha d(y)\beta y + x\alpha yk(\beta)y + x\alpha y\beta d(y)$$

$$= d(x)k(\alpha)d(y)\beta y + d(x)k(\alpha)yk(\beta)y + d(x)k(\alpha)y\beta d(y);$$

$$\Rightarrow (d(x)\alpha y + xk(\alpha)y + x\alpha d(y))\beta y + x\alpha y\beta d(y)$$

$$= (d(x)k(\alpha)d(y))\beta y + d(x)k(\alpha)y\beta d(y);$$

$$\Rightarrow d(x\alpha y)\beta y + x\alpha y\beta d(y) = d(x\alpha y)\beta y + d(x)k(\alpha)y\beta d(y);$$

$$\Rightarrow x\alpha y\beta d(y) = d(x)k(\alpha)y\beta d(y);$$

$$\Rightarrow (x\alpha y - d(x)k(\alpha)y)\beta d(y) = 0.$$

Lemma 2.2 *Let M be a semiprime* Γ_N *-ring, and let d be a k-derivation of M such that the associated mapping* $k:\Gamma \to \Gamma$ *is onto (= surjective), and* $M\Gamma M = M$. *If (a)* $xk(\alpha)x = 0$, *(b)* $d(x)k(\alpha)d(x) = 0$ *and (c)* $d(x)\alpha x = 0$ *hold for every* $x \in M$ *and* $\alpha \in \Gamma$, *then* $d = 0$.

Proof. Linearizing (b) on *x*, we get (for all $x, y \in M$ and $\alpha \in \Gamma$):

$$d(x+y)k(\alpha)d(x+y) = 0;$$

$$\Rightarrow d(x)k(\alpha)d(x) + d(x)k(\alpha)d(y) + d(y)k(\alpha)d(x) + d(y)k(\alpha)d(y) = 0;$$

$$\Rightarrow d(x)k(\alpha)d(y) + d(y)k(\alpha)d(x) = 0. \quad (2)$$

And, linearizing (c) on *x*, we have (for all $x, y \in M$ and $\alpha \in \Gamma$):

$$d(x+y)\alpha(x+y)=0;$$
$$\Rightarrow\ d(x)\alpha x+d(x)\alpha y+d(y)\alpha x+d(y)\alpha y=0;$$
$$\Rightarrow\ d(x)\alpha y+d(y)\alpha x=0;$$
$$\Rightarrow\ d(x)\alpha y=-d(y)\alpha x. \tag{3}$$

Let $m \in M$ and $\beta \in \Gamma$. By putting $m\beta x$ for *y* in (2), and then using equation (3) and the hypothesis (b) there, we obtain

$$d(x)k(\alpha)d(m\beta x)+d(m\beta x)k(\alpha)d(x)=0;$$
$$\Rightarrow\ d(x)k(\alpha)d(m)\beta x+d(x)k(\alpha)mk(\beta)x+d(x)k(\alpha)m\beta d(x)$$
$$+d(m)\beta xk(\alpha)d(x)+mk(\beta)xk(\alpha)d(x)+m\beta d(x)k(\alpha)d(x)=0;$$
$$\Rightarrow\ -d(x)k(\alpha)d(x)\beta m+d(x)k(\alpha)mk(\beta)x+d(x)k(\alpha)m\beta d(x)$$
$$-d(x)\beta mk(\alpha)d(x)+mk(\beta)xk(\alpha)d(x)=0;$$
$$\Rightarrow\ d(x)k(\alpha)mk(\beta)x+d(x)k(\alpha)m\beta d(x)$$
$$-d(x)\beta mk(\alpha)d(x)+mk(\beta)xk(\alpha)d(x)=0.$$

Again, let $\delta \in \Gamma$. Then we replace $x\delta x$ for *m* to get

$$d(x)k(\alpha)x\delta xk(\beta)x+d(x)k(\alpha)x\delta x\beta d(x)$$
$$-d(x)\beta x\delta xk(\alpha)d(x)+x\delta xk(\beta)xk(\alpha)d(x)=0.$$

Hence, by hypothesis, $d(x)k(\alpha)x\delta x\beta d(x)=0$. Since we assumed that $k:\Gamma\to\Gamma$ is onto, it produces $d(x)\Gamma M\Gamma M\Gamma d(x)=0$. But, since $M\Gamma M=M$, this yields $d(x)\Gamma M\Gamma d(x)=0$. Therefore, by the semiprimeness of *M*, it gives $d(x)=0$ for all $x \in M$, and we are done.

Theorem 2.1 *Let M be a semiprime* Γ_N *-ring. If d is a k-derivation of M acting as a k-endomorphism on M such that* $M\Gamma M=M$ *and* $xk(\alpha)x=0$ *hold for all* $x \in M$ *and* $\alpha \in \Gamma$*, then* $d=0$.

Proof. First, suppose *d* is a *k*-endomorphism on *M*. Applying Lemma 2.1(a) with $U=M$, it gives $d(x)\beta(x\alpha y-xk(\alpha)d(y))=0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Putting *y* by $y\delta m$ (for arbitrary $m \in M$ and $\delta \in \Gamma$), this yields

$$d(x)\beta(x\alpha y\delta m-xk(\alpha)d(y\delta m))=0;$$
$$\Rightarrow\ d(x)\beta(x\alpha y\delta m-xk(\alpha)d(y)\delta m-xk(\alpha)yk(\delta)m-xk(\alpha)y\delta d(m))=0;$$
$$\Rightarrow\ d(x)\beta(x\alpha y-xk(\alpha)d(y))\delta m$$
$$-d(x)\beta xk(\alpha)yk(\delta)m-d(x)\beta xk(\alpha)y\delta d(m)=0;$$
$$\Rightarrow\ -d(x)\beta xk(\alpha)yk(\delta)m-d(x)\beta xk(\alpha)y\delta d(m)=0;$$

$\Rightarrow\ d(x)\beta xk(\alpha)(yk(\delta)m+y\delta d(m))=0$.

Let $\mu\in\Gamma$ and replace $m\mu m$ for m in the last equation to get

$d(x)\beta xk(\alpha)(yk(\delta)m\mu m+y\delta d(m\mu m))=0$;

$\Rightarrow\ d(x)\beta xk(\alpha)(yk(\delta)m\mu m+y\delta d(m)\mu m+y\delta mk(\mu)m+y\delta m\mu d(m))=0$;

$\Rightarrow\ d(x)\beta xk(\alpha)(yk(\delta)m+y\delta d(m))\mu m+d(x)\beta xk(\alpha)y\delta m\mu d(m)=0$;

$\Rightarrow\ d(x)\beta xk(\alpha)y\delta m\mu d(m)=0$.

Therefore, it follows that $d(x)\beta xk(\alpha)y\delta M\mu d(M)=0$.

In particular, we have $d(x)\beta xk(\alpha)y\delta M\mu d(x)=0$.

This implies, $d(x)\beta xk(\alpha)y\delta M\mu d(x)\beta x=0$.

By definition, since $k:\Gamma\rightarrow\Gamma$ is onto, $(d(x)\beta x)\Gamma M\Gamma M\Gamma(d(x)\beta x)=0$.

Since $M\Gamma M=M$, it gives $(d(x)\beta x)\Gamma M\Gamma(d(x)\beta x)=0$.

Hence, by the semiprimeness of *M*, we obtain (for all $x\in M$ and $\beta\in\Gamma$)

$$d(x)\beta x=0 . \qquad (4)$$

Now, by taking Lemma 2.1(b) in the similar way, we have

$(x\alpha y-d(x)k(\alpha)y)\beta d(y)=0$ for all $x,y\in M$ and $\alpha,\beta\in\Gamma$.

Putting $m\delta x$ for *x* (for arbitrary $m\in M$ and $\delta\in\Gamma$), it gives

$(m\delta x\alpha y-d(m\delta x)k(\alpha)y)\beta d(y)=0$;

$\Rightarrow\ (m\delta x\alpha y-d(m)\delta xk(\alpha)y-mk(\delta)xk(\alpha)y-m\delta d(x)k(\alpha)y)\beta d(y)=0$;

$\Rightarrow\ m\delta(x\alpha y-d(x)k(\alpha)y)\beta d(y)$

$-d(m)\delta xk(\alpha)y\beta d(y)-mk(\delta)xk(\alpha)y\beta d(y)=0$;

$\Rightarrow\ -d(m)\delta xk(\alpha)y\beta d(y)-mk(\delta)xk(\alpha)y\beta d(y)=0$;

$\Rightarrow\ (d(m)\delta x+mk(\delta)x)k(\alpha)y\beta d(y)=0$.

Then by replacing $m\mu m$ for *m* (where $\mu\in\Gamma$), we get

$(d(m\mu m)\delta x+m\mu mk(\delta)x)k(\alpha)y\beta d(y)=0$;

$\Rightarrow\ (d(m)\mu m\delta x+mk(\mu)m\delta x+m\mu d(m)\delta x+m\mu mk(\delta)x)k(\alpha)y\beta d(y)=0$;

$\Rightarrow\ d(m)\mu m\delta xk(\alpha)y\beta d(y)+m\mu(d(m)\delta x+mk(\delta)x)k(\alpha)y\beta d(y)=0$;

$\Rightarrow\ d(m)\mu m\delta xk(\alpha)y\beta d(y)=0$.

Hence, this yields $d(M)\mu M\delta xk(\alpha)y\beta d(y)=0$.

In particular, we have $d(y)\mu M\delta xk(\alpha)y\beta d(y)=0$.

Then we get $y\beta d(y)\mu M\delta xk(\alpha)y\beta d(y)=0$.

Thus, we have $y\beta d(y)\Gamma M\Gamma M\Gamma y\beta d(y)=0$ (since $k:\Gamma\to\Gamma$ is onto here).

But, since $M\Gamma M = M$, it gives $(y\beta d(y))\Gamma M\Gamma(y\beta d(y))=0$.

So, by the semiprimeness of *M*, we obtain (for all $y\in M$ and $\beta\in\Gamma$)

$$y\beta d(y)=0. \tag{5}$$

Finally, we have $d(x\alpha y)=d(x)\alpha y+xk(\alpha)y+x\alpha d(y)$.

Putting $y=x$ here, we get $d(x\alpha x)=d(x)\alpha x+xk(\alpha)x+x\alpha d(x)$.

Hence, by using (4), (5) and (a), we obtain $d(x\alpha x)=0$; that is, for all $x\in M$ and $\alpha\in\Gamma$, we get

$$d(x)k(\alpha)d(x)=0. \tag{6}$$

Thus, all the conditions in the hypothesis of Lemma 2.2 are satisfied, and therefore, we obtain $d=0$. The proof is thus completed.

3. *k*-derivation acting as an anti-*k*-endomorphism

Definition 3.1 *Let M be a* Γ_N *-ring, and let* $k:\Gamma\to\Gamma$ *be an additive surjective mapping. Then a k-derivation* $d:M\to M$ *is said to be acting as an anti-k-homomorphism on M (which means, d acts as an anti-k-homomorphism of M onto M) if and only if* $d(a\alpha b)$ $=d(a)\alpha b+ak(\alpha)b+a\alpha d(b)=d(b)k(\alpha)d(a)$ *holds for all* $a,b\in M$ *and* $\alpha\in\Gamma$.

Lemma 3.1 *Let I be a right ideal of a* Γ_N *-ring M, and let d be a k-derivation of M acting as an anti-k-homomorphism on I such that* $xk(\alpha)x=0$ *for every* $x\in I$ *and* $\alpha\in\Gamma$. *Then, for all* $x,y\in I$, $m\in M$ *and* $\alpha\in\Gamma$ *:*

$$d(x)\alpha xk(\alpha)yk(\alpha)[m,d(x)]_{k(\alpha)}=0.$$

Proof. Since *d* acts as an anti-*k*-homomorphism on *I*, therefore, for all $x,y\in I$ and $\alpha,\beta\in\Gamma$, we have

$$d(x\alpha y)=d(x)\alpha y+xk(\alpha)y+x\alpha d(y)=d(y)k(\alpha)d(x). \tag{7}$$

Let $z\in M$ so that $x\alpha z\in I$ (since *I* is a right ideal of *M*). Then, by putting $x\alpha z$ for *y* in (7), we get

$$d(x)\alpha x\alpha z+xk(\alpha)x\alpha z+x\alpha d(x\alpha z)=d(x\alpha z)k(\alpha)d(x);$$

$$\Rightarrow\ d(x)\alpha x\alpha z+x\alpha d(z)k(\alpha)d(x)=d(x)\alpha zk(\alpha)d(x)$$

$$\Rightarrow\ +xk(\alpha)zk(\alpha)d(x)+x\alpha d(z)k(\alpha)d(x);$$

$$\Rightarrow\ d(x)\alpha x\alpha z=d(x)\alpha zk(\alpha)d(x)+xk(\alpha)zk(\alpha)d(x). \tag{8}$$

Next, replacing *z* by $xk(\alpha)y$ in (8), we obtain

$$d(x)\alpha x\alpha xk(\alpha)y=d(x)\alpha xk(\alpha)yk(\alpha)d(x)+xk(\alpha)xk(\alpha)yk(\alpha)d(x);$$

$$\Rightarrow\ d(x)\alpha x\alpha xk(\alpha)y=d(x)\alpha xk(\alpha)yk(\alpha)d(x). \tag{9}$$

Again, let $m \in M$ for which $yk(\alpha)m \in I$, since I is a right ideal of M. Then, by putting $yk(\alpha)m$ for y in (9), we get

$$d(x)\alpha x\alpha xk(\alpha)yk(\alpha)m = d(x)\alpha xk(\alpha)yk(\alpha)mk(\alpha)d(x). \quad (10)$$

Now, from (9), we have

$$d(x)\alpha x\alpha xk(\alpha)yk(\alpha)m = d(x)\alpha xk(\alpha)yk(\alpha)d(x)k(\alpha)m. \quad (11)$$

Comparing (10) and (11), for all $x, y \in I$, $m \in M$ and $\alpha \in \Gamma$, we get

$$d(x)\alpha xk(\alpha)yk(\alpha)mk(\alpha)d(x) = d(x)\alpha xk(\alpha)yk(\alpha)d(x)k(\alpha)m;$$

$$\Rightarrow d(x)\alpha xk(\alpha)yk(\alpha)(mk(\alpha)d(x) - d(x)k(\alpha)m) = 0;$$

$$\Rightarrow d(x)\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)} = 0.$$

Theorem 3.1 *Let M be a semiprime* Γ_N*-ring. If d is a k-derivation of M acting as an anti-k-endomorphism on M such that* $M\Gamma M = M$, $xk(\alpha)x = 0$ *and* $k(\alpha)x\alpha = \alpha xk(\alpha)$ *hold for all* $x \in M$ *and* $\alpha \in \Gamma$, *then* $d = 0$.

Proof. According to the hypothesis, by taking $I = M$ in Lemma 3.1, for all $x, y, m \in M$ and $\alpha \in \Gamma$, we have

$$d(x)\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)} = 0. \quad (12)$$

Replacing y by $mk(\alpha)y$ in (12), we get

$$d(x)\alpha xk(\alpha)mk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)} = 0. \quad (13)$$

Now, linearizing $xk(\alpha)x = 0$ on x, we have

$$(x + y)k(\alpha)(x + y) = 0;$$

$$\Rightarrow xk(\alpha)x + xk(\alpha)y + yk(\alpha)x + yk(\alpha)y = 0;$$

$$\Rightarrow xk(\alpha)y + yk(\alpha)x = 0;$$

$$\Rightarrow xk(\alpha)y = -yk(\alpha)x. \quad (14)$$

Then, by using the hypothesis along with (12), (13) and (14), we get

$$[m, d(x)]_{k(\alpha)}\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)};$$

$$= (mk(\alpha)d(x) - d(x)k(\alpha)m)\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)};$$

$$= mk(\alpha)d(x)\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)}$$

$$- d(x)k(\alpha)m\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)};$$

$$= mk(\alpha)(d(x)\alpha xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)})$$

$$- d(x)\alpha mk(\alpha)xk(\alpha)yk(\alpha)[m, d(x)]_{k(\alpha)};$$

$$= 0 + d(x)\alpha x k(\alpha) m k(\alpha) y k(\alpha)[m, d(x)]_{k(\alpha)} = 0.$$

Here, since $k:\Gamma \to \Gamma$ is to be considered as onto, this yields

$$[m, d(x)]_{k(\alpha)}\Gamma M\Gamma M\Gamma[m, d(x)]_{k(\alpha)} = 0.$$

As $M\Gamma M = M$, it gives $[m, d(x)]_{k(\alpha)}\Gamma M\Gamma[m, d(x)]_{k(\alpha)} = 0$. But, since M is semiprime, we get $[m, d(x)]_{k(\alpha)} = 0$ for all $x, m \in M$ and $\alpha \in \Gamma$. Hence, it follows that $mk(\alpha)d(x)$ $= d(x)k(\alpha)m$, and therefore, $d(x) \in C_{k(\alpha)}$ for all $x \in M$ and $\alpha \in \Gamma$. So, we obtain $d(x) \in C_{\Gamma}$ (since $k:\Gamma \to \Gamma$ is onto).

Hence, we get $d(x\alpha y) = d(y)k(\alpha)d(x) = d(x)k(\alpha)d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. So, by definition, d is then a *k*-endomorphism on *M*. Therefore, by Theorem 2.1, it follows that $d = 0$. This completes the proof.

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