GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) **34** (2014) 47-55

SOME CURVATURE PROPERTIES IN LP-SASAKIAN MANIFOLDS WITH RESPECT TO GENERALIZED TANAKA WEBSTER OKUMURA CONNECTION

Ali Akbar[1], A. Sarkar[2] and Md. Showkat Ali[3]

[1]*Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India*

[2]*Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India*

[3]*Department of Applied Mathematics, University of Dhaka, Dhaka-1000, Bangladesh*

[1]Email: aliakbar.akbar@rediffmail.com; [2]Email: avjaj@yahoo.co.in; [3]Email: msa317@yahoo.com

Received 05.05.2014 Accepted 07.12.2014

ABSTRACT

The object of the present paper is to study LP-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We have studied locally ϕ-symmetric as well as locally projectively ϕ-symmetric LP-Sasakian manifolds with respect to a generalized Tanaka Webster Okumura connection. Locally ϕ-recurrent LP-Sasakian manifolds have also been studied with respect to generalized Tanaka Webster Okumura connection.

Keywords: Curvature Tensor, LP-Sasakian Manifolds, Tanaka Webster Okumura Connection

1. Introduction

For a differential geometer most important topic of study is symmetry of manifolds. The famous geometer E. Cartan [2], [3], pioneered the study of symmetry on manifolds. According to Cartan, a differentiable manifold M is called locally symmetric [2], [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor and ∇ is the Levi-Civita connection on the manifold. The symmetry of a manifold has been weakened by several authors in several ways. For instance, in 1977 Takahashi [15] introduced the notion of local ϕ-symmetry on Sasakian manifolds. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X, Y)Z = 0 \tag{1.1}$$

for any vector fields X, Y, Z, W orthogonal to ξ, where ξ is the structure vector field and ϕ is the structure tensor on the manifold M. The concept of local ϕ-symmetry on various structures and their generalizations or extensions have been studied by several authors [4], [5], [11], [12], [13], [14]). In ([5], De, Shaikh and Biswas generalized the notion of local ϕ-symmetry and introduced the notion of ϕ-recurrent Sasakian manifolds.

There is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, Matsumoto [17] introduced the notion of Lorentzian para-Sasakian manifolds. Again, Mihai and Rosaca [6] introduced the same notion independently and obtained many interesting results. Lorentzian para-Sasakian manifolds are briefly known as LP-Sasakian manifolds. LP-Sasakian manifolds have also been studied by De, Matsumoto and Shaikh [18], Mihai, De and Shaikh [7], Shaikh and Baishya [10] and several others. The notion of generalized Tanaka Webster Okumura connection has been introduced and studied by Inoguchi and Lee [8]. Two of the present authors studied trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection [1], [9]. In the present paper we study LP-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. The present paper is organized as follows:

After introduction in Section 1 we give some preliminaries in Section 2. In Section 3 we establish a relation between the curvature tensor $\bar{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\bar{\nabla}$ and the Levi-Civita connection ∇ respectively in an LP-Sasakian manifold. In Sections 4 and 5 we have studied respectively locally ϕ-symmetric and locally projectively ϕ-symmetric LP- Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Section 6 is completed with the study of locally ϕ-recurrent LP-Sasakian manifolds with respect to a generalized Tanaka Webster Okumura connection.

2. Preliminaries

An n-dimensional, differentiable manifold M^n is called Lorentzian para-Sasakian manifold [16], [17] if it admits (1, 1)-tensor field ϕ, a contravariant vector field ξ, 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1 \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.3}$$

$$g(X,\xi) = \eta(X) \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where ∇ denotes the covariant differentiation with respect to Lorentzian metric. It can be easily seen that in an LP-Sasakian manifold the following relations hold.

$$\phi\xi = 0,\ \eta(\phi) = 0, \tag{2.7}$$

$$\text{rank}\ (\phi) = n - 1. \tag{2.8}$$

If we put

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.9}$$

for any vector field X, Y, then the tensor field $\Phi(X, Y)$ is a symmetric (0,2)-tensor field [16]. Also since the 1-form η is closed in an LP –Sasakian manifold, we have [17], [18]

$$(\nabla_X \eta)(Y) = \Phi(X, Y),\ \Phi(X, \xi) = 0 \tag{2.10}$$

for all X, Y $\in$ TM.

Also in an LP-Sasakian manifold, the following relations hold [16], [18]

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y , \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\, \xi - \eta(Y)X , \tag{2.12}$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z)=g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n - 1)\eta (X), \tag{2.15}$$

$$g(X, \xi) = \eta(X) \tag{2.16}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{2.17}$$

where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold respectively.

3. Curvature tensor of an LP-Sasakian manifold with respect to the generalized Tanaka Webster Okumura connection

In an LP-Sasakian manifold the generalized Tanaka Webster Okumura connection [8] $\overline{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\overline{\nabla}_X Y= \nabla_X Y + A(X, Y) \tag{3.1}$$

for all vectors fields X, Y on M and

$$A(X, Y) = - \{g(X, \phi Y)\xi + \eta(Y)\phi X\} - \ell\eta(X)\phi(Y). \tag{3.2}$$

where ℓ is a real constant.

Now applying $\overline{\nabla}$ on both sides of (3.1) and by straightforward calculation we get,

$$\begin{aligned}\overline{\nabla}_Y \overline{\nabla}_X Z = {} & \nabla_Y\nabla_X Z - g(Y, \phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \ell\eta(Y)\phi\nabla_X Z \\ & - g(\nabla_Y X, \phi Z)\xi - g(X,\nabla_Y\phi Z)\xi \\ & -\{g(A(Y,X),\phi Z) + g(X,A(Y,\phi Z))\}\xi \\ & - g(X,\phi Z)\{\nabla_Y\xi - \eta(\xi)\phi Y\} \\ & -\eta(Z)\, \{\nabla_Y\phi X + g(\phi X,\phi Y\}\xi - \ell\eta(Y)\phi^2 X\} \\ & - \ell\eta(X)\, \{\nabla_Y\phi X + g(\phi X,\phi Y\}\xi - \ell\eta(Y)\phi^2 X\} \\ & -\{\nabla_Y\eta(Z) + A(Y,\eta(Z))\}\phi x - \ell\{\nabla_Y\eta(X) + A(Y,\eta(X))\}\phi Z \end{aligned} \tag{3.3}$$

Interchanging X and Y in (3.3) we get,

$$\begin{aligned}\overline{\nabla}_X \overline{\nabla}_Y Z = {} & \nabla_X\nabla_Y Z - g(X, \phi\nabla_Y Z)\xi - \eta(\nabla_Y Z)\phi X - \ell\eta(X)\phi\nabla_Y Z \\ & - g(\nabla_X Y, \phi Z)\xi - g(Y,\nabla_X\phi Z)\xi \\ & -\{g(A(X,Y),\phi Z) + g(Y,A(X,\phi Z))\}\xi \\ & - g(Y,\phi Z)\{\nabla_X\xi - \eta(\xi)\phi X\} \end{aligned}$$

$$-\eta(Z)\,\{\nabla_X\phi Y + g(\phi Y, \phi X)\xi - \ell\eta(X)\phi^2 Y\}$$
$$-\ell\eta(Y)\,\{\nabla_X\phi Y + g(\phi Y, \phi X)\xi - \ell\eta(X)\phi^2 Y\}$$
$$-\{\nabla_X\eta(Z) + A(X,\eta(Z))\}\phi Y - \ell\{\nabla_X\eta(Y) + A(Y, \eta(Y))\}\phi Z \qquad (3.4)$$

Also by using (3.1) we have

$$\overline{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z - \{g([X,Y], \phi Z)\xi + \eta(Z)\phi[X,Y]\} - \ell\eta([X,Y])\phi Z \qquad (3.5)$$

We know that

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad (3.6)$$

and

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z \qquad (3.7)$$

where $\overline{R}$ and R are the Riemannian curvature tensor with respect to $\overline{\nabla}$ and ∇ respectively. After using (3.3), (3.4) and (3.6) in (3.7) and by straight forward calculation we get,

$$\overline{R}(X, Y)Z = R(X, Y)Z - \{g(Y, \nabla_X\phi Z)\xi - g(X, \nabla_Y\phi Z)\xi + g(X, \phi\nabla_Y Z)\xi$$
$$+ \eta(\nabla_Y Z)\phi X - g(Y,\phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi[X, Y]\}$$
$$-\ell\{\eta(X)\phi\nabla_Y Z - \eta(Y)\phi\nabla_X Z - \eta([X, Y])\phi Z\}$$
$$- g(Y, \phi Z)\{\nabla_X\xi - \eta(\xi)\phi X\} + g(X, \phi Z)\{\nabla_Y\xi - \eta(\xi)\phi Y\}$$
$$+\{g(A(Y, X), \phi Z) + g(X, A(Y, \phi Z) - g(A, (X, Y), \phi Z) - g(Y,A(X, \phi Z))\}\xi$$
$$+ \eta(Z)\{\nabla_Y\phi X - \ell\eta(Y)\phi^2 X - \nabla_X\phi Y + \ell\eta(X)\phi^2 Y + \phi[X,Y]$$
$$+ \ell\eta(X)\{\nabla_Y\phi Z + g(\phi Y, \phi Z)\xi - \ell\eta(Y)\,\phi^2 Z\}$$
$$-\ell\eta(Y)\{\nabla_X\phi Z + g(\phi X, \phi Z)\xi - \ell\eta(X)\,\phi^2 Z\}$$
$$+ \{\nabla_Y\eta(Z) + A(Y,\eta(Z))\}\phi X + \ell\{\nabla_Y\eta(X) + A(Y, \eta(X))\}\phi Z$$
$$- \{\nabla_X\eta(Z) + A(X, \eta(Z))\}\phi Y - \ell\{\nabla_X\eta(Y) + A(X, \eta(Y))\}\phi Z \qquad (3.8)$$

We suppose that X, Y, Z, $\nabla_X Z$ and $\nabla_Y Z$ are orthogonal to ξ. Then (3.8) becomes

$$\overline{R}(X, Y)Z = R(X, Y)Z - \{g(Y, \nabla_X\phi Z) - g(X, \nabla_Y\phi Z) + g(X, \phi\nabla_Y Z) - g(Y, \phi\nabla_X Z)\}\xi$$
$$- g(Y, \phi Z)\{\nabla_X\xi - \eta(\xi)\phi X\} + g(X, \phi Z)\{\nabla_Y\xi - \eta(\xi)\phi Y\}. \qquad (3.9)$$

Using (2.1) and (2.5) we rearrange (3.9) as

$$\overline{R}(X, Y)Z = R(X, Y)Z + \{g(Y, \phi\nabla_X Z) - g(X, \phi\nabla_Y Z)\}\xi$$
$$+\{g(X, \nabla_Y\phi Z) - g(Y, \nabla_X\phi Z)\}\xi + 2g(X, \phi Z)\phi Y - 2g(Y, \phi Z)\phi X \qquad (3.10)$$

From equation (3.10) it follows that

$$S(X, Y) = S(X, Y) + 2g(X, Y) \qquad (3.11)$$

$$QX = QX + 2X, \tag{3.12}$$

and

$$\bar{r} = r + 2(n+1) \tag{3.13}$$

where $\bar{S}$, S, $\bar{Q}$, Q, $\bar{r}$ and r are corresponding Ricci tensors, Ricci operators and Scalar curvature of the LP-Sasakian manifold with respect to generalized Tanaka Webster Okumura connection and Levi-Civita connection respectively.

4. Locally ¤-symmetric LP-Sasakian manifolds with respect to the Generalized Tanaka Webster Okumura Connection

Definition 4.1. An LP-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X, Y)Z = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds [15].

Analogous to the definition of locally ϕ-symmetric LP-Sasakian manifolds with respect to Levi-Civita connection, we define locally ϕ-symmetric LP-Sasakian manifold with respect to generalized Tanaka Webster Okumura connection by

$$\phi^2(\bar{\nabla}_W \bar{R})(X, Y)Z = 0, \tag{4.2}$$

for all vector fields X, Y, Z and W orthogonal to ξ.

Now, differentiating both sides covariantly by W with respect to the Levi-Civita connection ∇ we obtain from (3.10)

$$\begin{aligned}\phi^2(\bar{\nabla}_W \bar{R})(X, Y)Z &= (\nabla_W R)(X, Y)Z + \{g(Y, (\nabla_W\phi)\nabla_X Z) - g(X, (\nabla_W\phi)\nabla_Y Z)\xi \\ &+ \{g(Y, \phi\nabla_X Z) - g(X, \phi\nabla_Y Z) + g(X, \nabla_Y\phi Z) - g(Y, \nabla_X\phi Z)\nabla_W\xi \\ &+ 2g(X, (\nabla_W\phi)Z)\phi Y - 2g(Y, (\nabla_W\phi)Z)\phi X) + 2g(X, \phi Z)(\nabla_W\phi)Y \\ &- 2g(Y, \phi Z)(\nabla_W\phi)X \end{aligned} \tag{4.3}$$

Using (2.5) and (2.6) and considering W orthogonal to ξ we obtain from (4.3)

$$\begin{aligned}(\bar{\nabla}_W \bar{R}))(X, Y)Z &= (\nabla_W R)(X, Y)Z + \{g(Y, \phi\nabla_X Z) - g(X, \phi\nabla_Y Z) + g(X, \nabla_Y\phi Z) \\ &- g(Y, \nabla_X\phi Z)\phi W + 2g(X, \phi Z)g(W, Y)\xi \\ &- 2g(Y, \phi Z)g(W, X)\xi \end{aligned} \tag{4.4}$$

We suppose that X and Y are orthogonal to ξ. Then (3.1) and (3.2) give

$$\bar{\nabla}_X Y = \nabla_X Y - g(X, \phi Z)\xi. \tag{4.5}$$

Using (4.5) we have

$$(\bar{\nabla}_W \bar{R})((X, Y)Z) = (\bar{\nabla}_W \bar{R})((X, Y)Z) - g(W, \phi \bar{R}((X, Y)Z)\xi. \tag{4.6}$$

In view of (4.4) we obtain from (4.6)

$$(\overline{\nabla}_W \overline{R}))(X, Y)Z = (\nabla_W R)(X, Y)Z + \{g(Y, \phi\nabla_X Z) - g(X, \phi\nabla_Y Z)\xi + g(X, \nabla_Y \phi Z) - g(Y, \nabla_X \phi Z)\}\phi W + 2g(X, \phi Z)g(W, Y)\xi - 2g(Y, \phi Z)\, g(W, X)\xi - g(W, \phi R(X, Y)X)\xi . \quad (4.7)$$

Applying ϕ^2 on both sides of (4.7) and using (2.2) and (2.7) we get

$$\phi^2 (\overline{\nabla}_W \overline{R}))(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z + \{g(Y, \phi\nabla_X Z) - g(X, \phi\nabla_Y Z)\xi + g(X,\nabla_Y \phi Z) - g(Y, \nabla_X \phi Z)\}\phi W. \quad (4.8)$$

Taking inner product with respect W on both sides of (4.8) we get

$$g(\phi^2((\overline{\nabla}_W \overline{R}))((X, Y)Z, W) = g\, \phi^2(\nabla_W R)((X, Y)Z, W). \quad (4.9)$$

The relation (4.9) is true for any vector field W. So we can write

$$\phi^2 (\overline{\nabla}_W \overline{R})(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z, \quad (4.10)$$

for vector fields X, Y, Z and W orthogonal to ξ. Thus we are in a position to state the following.

Theorem 4.1. An LP-Sasakian manifold is locally ϕ symmetric with respect to generalized Tanaka Webster Okumura connection if and only if it is so with respect to Levi-Civita connection.

5. Locally projectively ¤-symmetric LP-Sasakian manifolds with respect to generalized Tanaka Webster Okumura Connection

In this section we like to study locally projectively ϕ-symmetric LP-Sasakian manifolds with respect to generalized Tanaka Webster Okumura Connection. Let us first recall the following well known definition.

Definition 5.1 An LP-Sasakian manifold M^n is said to be locally projectively ϕ-symmetric if

$$¤^2(ď_W P)(X, Y)Z = 0 \quad (5.1)$$

for all vector fields X,Y,Z,W orthogonal to ξ , P is the projective curvature tensor defined by

$$P(X, Y)Z = R(X, Y)Z - 1/(n-1)\{S(Y, Z)X - S(X, Z)Y\} \quad (5.2)$$

Differentiating covariantly by W with respect to the Levi-Civita connection ∇ on both sides of (5.2) we obtain

$$(\overline{\nabla}_W P)(X, Y)Z = (\overline{\nabla}_W R)(X, Y)Z - 1/(n-1)\{\overline{\nabla}_W S(Y, Z)X - \overline{\nabla}_W S(X, Z)Y\} \quad (5.3)$$

Definition 5.2. Analogous to the definition to locally ϕ-symmetric LP-Sasakian manifolds with respect to Levi-Civita connection, we define locally ϕ-symmetric LP-Sasakian manifold with respect to generalized Tanaka Webster Okumura connection by

$$\phi^2(\overline{\nabla}_W \overline{P})(X, Y)Z = 0 \quad (5.4)$$

for all vector fields X, Y, Z and W orthogonal to ξ and $\overline{P}$ is the projective curvature tensor defined by

$$\overline{P}(X, Y)Z = \overline{R}(X, Y)Z - 1/(n-1)\{\overline{S}(Y, Z)X - \overline{S}(X, Z)Y\} \quad (5.5)$$

where $\overline{R}$ and $\overline{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to generalized Tanaka Webster Okumura connection.

Now using (4.7) we obtain

$$(\overline{\nabla}_W \overline{P})(X, Y)Z = (\nabla_W \overline{P})(X,Y)Z - g(W,¤\ \overline{P}((X, Y)Z)\xi. \quad (5.6)$$

Differentiating covariantly by W with respect to the generalized Tanaka Webster Okumura connection $\overline{\nabla}$ on both sides of (5.5) we obtain

$$(\overline{\nabla}_W \overline{P})(X, Y)Z = (\nabla_W \overline{R})(X, Y)Z - 1/(n-1)\{(\nabla_W \overline{S})(Y, Z)X - (\nabla_W \overline{S})(X, Z)Y\} \quad (5.7)$$

In view of (3.11) and (4.4), (5.7) reduces to

(∇̄_W P̄)(X, Y)Z = (ď_W R) (X, Y)Z + {g(Y, ¤ď_X Z) J g(X, ¤ď_Y Z) + g(X, ď_Y ¤Z)
J g(Y, ď_X ¤Z)¤W + 2g(X, ¤Z)g(W, Y)¶ J 2g(Y, ¤Z)g(W, X) ¶
$-1/(n-1)\{(\nabla_W \overline{S})(Y, Z)X - (\nabla_W \overline{S})(X, Z)Y\}$ (5.8)

Using (5.3) in (5.8) we get

(∇̄_W P̄)(X, Y)Z= (ď_W R)(X, Y)Z + {g(Y, ¤ď_X Z) J g(X, ¤ď_Y Z)+g(X, ď_Y ¤Z)
J g(Y, ď_X ¤Z)}¤W+2g(X, ¤Z)g(W, Y)¶J2g(Y, ¤Z)g(W, X)¶ (5.9)

In view of (5.9) we obtain from (5.6)

(∇̄_W P̄) (X, Y)Z = (ď_W R) (X, Y)Z + {g(Y, ¤ď_X Z) J g(X, ¤ď_Y Z)+g(X, ď_Y ¤Z)
J g(Y,ď_X ¤Z)}¤W+{2g(X, ¤Z)g(W, Y)J2g(Y, ¤Z)g(W, X)
J g(W, ¤_W P̄ (X, Y)X)}¶. (5.10)

Applying ϕ^2 on both side of (5.10) and using (2.2) and (2.7) we get

ϕ^2(∇̄_W P̄)(X, Y)Z = ¤²(ď_W P) (X, Y)Z + {g(Y, ¤ď_X Z) J g(X, ¤ď_Y Z)
+g(X,ď_Y ¤Z) J g(Y, ď_X ¤Z)}¤W (5.11)

Taking inner product with respect to W on both sides of (5.11) we get

$$g(\phi^2(\overline{\nabla}_W \overline{P})(X, Y)Z, W) = g(\phi^2(\nabla_W P)(X, Y)Z), W). \quad (5.12)$$

The relation (5.12) is true for any vector field. So we can write

$$\phi^2(\overline{\nabla}_W \overline{P})(X, Y)Z = \phi^2(\nabla_W P)(X, Y)Z), \quad (5.13)$$

for any vector fields X, Y, Z and W orthogonal to ξ. Thus we are in a position to state the following:

Theorem 5.1. : An LP-Sasakian manifold is locally projectively ϕ-symmetric with respect to generalized Tanaka Webster Okumura connection if and only if it is so with respect to Levi-Civita connection.

6. Locally ¤- recurrent LP-Sasakian manifolds with respect to generalized Tanaka Webster Okumura Connection

Definition 6.1. An LP-Sasakian Manifold will be called locally ϕ- recurrent with respect to Levi-Civita connection ∇ if

$$¤^2(\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z. \tag{6.1}$$

for the vector fields X, Y, Z and W orthogonal to ξ , A is an 1-form defined by A(W) = g(W,ρ), for some vector field ρ. In this connection it should be mentioned that the notion of locally ϕ-recurrent manifolds was introduced in the paper [5] in the context of Sasakian Geometry.

Definition 6.2. An LP-Sasakian Manifold will be called locally ϕ- recurrent with respect to the generalized Tanaka Webster Okumura connection $\overline{\nabla}$ if

$$¤^2(\overline{\nabla}_W \overline{\nabla})(X, Y)Z = A(W)\overline{R}(X, Y)Z. \tag{6.2}$$

for any vector fields X, Y, Z and W orthogonal to ξ and A is an 1-form defined by A(W) = g(W,ρ), for some vector field ρ.

In view of (2.2) we obtain from (6.2)

$$g((\overline{\nabla}_W \overline{R})(X, Y)Z, W) = A(W)g(\overline{R}(X, Y)Z, W). \tag{6.3}$$

Using (3.1) and (4.7) in (6.3) we obtain

$$g((\nabla_W R)(X, Y)Z, W) = A(W)g(R(X, Y)Z, W) + 2g(X, \phi Z)g(\phi Y, W)$$
$$- 2g(Y, \phi Z)g(\phi X, W) \tag{6.4}$$

We choose W in such a way that A(W)=1 and setting X = Y in (6.4) we obtain

$$g((\nabla_W \overline{R})(X, Y)Z,W) = g(R(X, Y)Z, W) \tag{6.5}$$

Since W is any arbitrary vector field, so we obtain from (6.5)

$$(\nabla_W \overline{R})(X, Y)Z = R(X, Y)Z \tag{6.6}$$

Applying ϕ^2 on both side of (6.6) and using (2.2) and (2.13) we get

$$\phi^2(\nabla_W \overline{R})(X, Y)Z = A(W)R(X, Y)Z, \tag{6.7}$$

where X, Y, Z and W are vector fields orthogonal to ξ and A is an 1-form defined by A(W) = 1. Thus, we are in a position to state the following:

Theorem 6.1. : An Lp-Sasakian manifold is locally ϕ-recurrent with respect to generalized Tanaka Webster Okumura connection if and only if it is so with respect to Levi-Civita connection.

7. Conclusion

In the theory of differentiable manifolds, symmetry is an important property. Our results are the generalizations of the results of Takahashi [15] and De, Shaikh, Biswas [5] regarding symmetry and its generalizations. We show that an LP-Sasakian manifold is locally φ-symmetric with respect to generalized Tanaka Webster Okumura connection if and only if it is so with respect to Levi Civita connection. The fact is also true for projectively locally φ-symmetric manifolds and φ-recurrent manifolds.

Acknowledgement. The authors are thankful to the referee for valuable suggestions in the improvement of the paper.

REFERENCES

[1] Akbar, A. and Sarkar, A., On three dimensional locally ϕ-symmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. (submitted to *Lobachevskii J. Math*.)

[2] Cartan, E., Sur une class remarquable despace de Riemann, I, *Bull. Soc. Math. de France*, 54(1926), 214-216.

[3] Cartan, E., Sur une class remarquable despace de Riemann, II, *Bull, Soc. Math. de France*, 55(1927), 218-226.

[4] Özgür, C., ϕ- conformally flat Lorentzian para-Sasakian manifolds, *Radovi Mathematticki* 12(2003), 1- 8.

5] De, U. C., Shaikh, A. A. and Biswas, S., On ϕ-recurrent Sasakian manifolds, *Novi Sad J. Math.,* 22(2003), 43-48.

[6] Mihai, I. and Rosaca, R., On Lorentzian P-Sasakian manifold, Classical Analysis, World Scientific Publi., Singapore (1992),155-169.

[7] Mihai, I., De, U. C. and Shaikh, A. A., On Lorentzian para-Sasakian manifolds, *Korean J. Math. Sciences*, 6(1999) 1-13.

[8] Inoguchi, J. and Lee, J. E., Affine biharmonic curves in 3-dimensional homogenous geometries, *Mediterr. J. Math.*, 10(2013), 571-592.

[9] Sarkar, A. and Akbar A., Some curvatrure properties of trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumra connection (submitted).

[10] Shaikh, A. A. and Baishya, K. K., Some results on LP-Sasakian manifolds, *Bull, Math. Soc, Sci. Math. Mommanic*, 97(2006), 197-205.

[11] Shaikh, A. A., Baishya, K. K., On ϕ-symmetric LP-Sasakian manifolds, *Yokohama Math. J.,* 53(2005), 97-112.

[12] Shaikh, A. A., Baishya, K. K. and Eyasmin, S., On ϕ-recurrrent generalized (κ,μ)-contact metric manifolds, *Lobachevskii J. Math.,* 27 (2007), 3-13.

[13] Shaikh, A. A., Basu, T. and Eyasmin, S., On locally ϕ-symmetric $(LCS)_n$-manifolds, *Int. J. of Pure and Appl. Math*, 41(2007), 1161-1170.

[14] Shaikh, A. A., Basu, T. and Eyasmin, S., On the existence of ϕ-recurrent $(LCS)_n$- manifolds, *Extracta Mathematicae*, 23(1) (2008), 71-83.

[15] Takahashi, T. Sasakian ϕ-symmetric spaces, *Tohoku Math. J.*, 29 (1977), 91-113.

[16] Matsumoto, K. and Mihai, I., On certain transformation in a Lorentzian Para-Sasakian manifold, *Tensor*, N.S. 47 (1988), 189-197.

[17] Matsumoto, K., On Lorentzian almost paracontact manifolds, *Bull Yamagata Univ. Nat. Sci.* 12(1989), 151-156.

[18] De, U. C., Matsumoto, K. Shaikh, A. A., On Lorentzian para-Sasakian Manifolds, *Rendiconti del seminario mat, de Messina*, 3(1999), 149-156.