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ENDOMORPHISM RINGS ARE CENTRALIZER NEAR-RINGS

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ABSTRACT

For a finite ring R with identity and a finite unital R-module V the set $C(R; V) = \{f : V \to V : f(\alpha v) = \alpha f(v)$ for all $\alpha \in R$, $v \in V\}$ is the centralizer near-ring determined by R and V. Rings R for witch $C(R; V)$ is a ring for every R-module V, are characterized. Conditions are given under which $C(R; V)$ is a semisimple centralizer near ring. Its shown that $C(R; V)$ is a semisimple centralizer near ring then $End_R(V) = C(R; V)$.

Keywords: Centralizer Near-ring, Semisimple Centralizer Near-ring, $C(R, V)$ invariant

1. Introduction.

Let G be a group and T a semigroup of endomorphisms of G. Then

$C(T;G) = \{ f : G \to G : f(0) = 0$ and $f(¥a) =$ ª $f(a)$ for all ª $\in T, a \in G\}$ is a near-ring under the operations of addition and composition of functions, and is called the centralizer near-ring determined by T and G. It has been shown by Betsch [1] that N is a finite simple near-ring with identity if and only if there exists a finite group G and a fixed point free group of automorphism T of G such that $N \cong C(T;G)$. The structure of $C(T; G)$ for various G and T has been investigated by Maxson and Smith [3], [4], [5].

Throughout this paper R will denote a finite ring with 1 and V a finite unital R-module. The corresponding centralizer near-ring is $C(R;V) = \{ f : V \to V : f(rv) = r f(v)$ for all $r \in R$, $v \in V\}$. In dealing with $C(R; V)$ we may assume, without loss of generality, that V is a faithful R-module, because V is a faithful $\overline{R}$-module where $\overline{R} = R/\text{Ann}(V)$, and we have $C(R;V) = C(\overline{R}; V)$.

It is the goal of this paper to consider the following questions which arise naturally from the above remarks.

A. Which finite rings R have the property that $C(R; V)$ is a ring for every R-module V?

B. If $C(R; V)$ is a semisimple ring when is $C(R; V) = \text{End}_R(V)$?

C. Which semisimple near-rings have the form $C(R; V)$ for some pair (R, V)?

Subsequently we will answer question A. We also show that if $C(R; V)$ is a semisimple ring then one always has $C(R; V) = \text{End}_R(V)$. Moreover if $C(R; V)$ is semisimple then information about the structure of the simple components is obtained, giving a partial answer to question C. Subsequently we will show that if *C*(*R; V*) is a semisimple ring then $End_R(V) = C(R;V)$.

2. Semisimple Centralizer Near-ring

At first we will define semisimple centralizer near-ring. Then some characteristics or properties of semisimple centralizer near-ring will be established.

Definition 2.1. Semisimple Centralizer Near-rings [2] Let *C*(*R; V*) be semisimple. Then the center of *C*(*R; V*) cannot contain nonzero nilpotent elements. Hence the center of *R* cannot contain nilpotent elements so the center of *R* is a direct sum of fields. Thus if *n* is the characteristic of *R*, we have $n = p_1 p_2 ... p_s$ where p_i's are distinct primes. But this implies that $R = R_1 \oplus ... \oplus R_s$ where R_i has characteristic p_i. Because it has characteristic p_i, R_i is an algebra over the field *GF*(p_i) and so the Wedderburn principal theorem [7, p. 164] holds for R_i. Consequently $R = \sum_{ij} S_{ij} + N$ where each S_{ij} is a simple ring and *N* is a nilpotent ideal of *R*.

Proposition 2.1. [2]. Let *R* be a finite semisimple ring and let *V* be a finite *R*-module. Then *C*(*R; V*) is a semisimple near-ring.

Proof: We have $R = S_1 \oplus ... \oplus S_t$, where each S_i is a simple ring. Let e_i denote the identity of S_i. If $V_i = \{v \in V : e_i v = v\}$ then $V = V_1 \oplus ... \oplus V_t$, and $f(V_i) \subseteq V_i$ for each $f \in C(R;V)$. Further, if f_i denotes the restriction of f to V_i then the map ¤ $: C(R;V) \to C(S_1;V_1) \oplus ... \oplus C(S_t;V_t)$ given by ¤$(f) = \langle f_1, ..., f_t \rangle$ is a near-ring homomorphism. The map is onto, for if $\langle f_1, ..., f_t \rangle$ is in $C(S_1;V_1) \oplus ... \oplus C(S_t;V_t)$ extend each f_i to all of *V* by $\bar{f}_i(v_1 + ... + v_t) = f_i(v_i)$. Then $f = \sum \bar{f}_i$ is an element of *C*(*R; V*) such that ¤$(f) = \langle f_1, ..., f_t \rangle$. To show that ¤ is one-to-one we note that $e_i f(v_1 + ... + v_t) = f(e_i v_i) = f(v_i), i = 1, ..., t$ implies $f(v_1 + ... + v_t) = f(v_1) + ... + f(v_t)$ $= f_1(v_1) + ... + f_t(v_t)$. Hence if ¤$(f) = 0$ then $f = 0$. Therefore ¤ is an isomorphism and from Theorem 1of [6] each $C(S_i;V_i)$ is a simple near-ring.

Proposition 2.2. If *C*(*R; V*) is a semisimple near-ring for every *R*-module *V* then in particular *C*(*R; R*) is semisimple. But *C*(*R; R*) is anti-isomorphic to *R* so *R* is a semisimple ring.

Proof: If $R = S_1 \oplus ... \oplus S_t$, S_i simple and not a field, or S_i is a field and $\dim_{S_i}(V_i) = 1$, we have *C*(*R; V*) is a semisimple ring. Moreover, in this setting, $C(R;V) = End_R(V)$.

Theorem 2.1. *C*(*R; V*) is a semisimple ring which is not a ring.

Proof: We will prove this theorem with the help of an example. Let $R = \overline{R} \oplus F$ where F = *GF*(2) and $\overline{R}$ is the simple ring of 2 x 2 matrices over *GF*(2). Let

$V_i = \left\{ \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in F \right\}, i = 1,2, \right.$ and let R act on $V = V_1 \oplus V_2$ componentwise. Then $C(R;V) \cong C(\overline{R};V_1) \oplus C(F;V_2)$ where $C(\overline{R};V_1)$ is a simple ring while $C(F;V_2)$ is a simple near-ring which is not a ring. Hence $C(R; V)$ is semisimple and not a ring.

Theorem 2.2. When $C(R; V)$ is semisimple centralizer near-ring then $End_R(V) = C(R;V)$.

Proof: As we have seen $R = S_1 \oplus ... \oplus S_t + N$ where each S_i is simple and N is a nilpotent ideal of R. We may assume $N \neq (0)$; otherwise R is semisimple and the previous result applies.

Assume t = 1, i.e. $R = S_1 + N$. From the proof of Lemma 1 of [6] it follows that $C(R; V)$ contains a function f such that $g_1 f g_2 f = 0$ for all $g_1, g_2 \in C(R;V)$. Hence $C(R; V)$ contains a nilpotent $C(R; F)$-subgroup and is not semisimple. So we may assume t > 1.

Let e_i denote the identity for S_i . Then $V = V_1 \oplus ... \oplus V_t$ where $V_i = \{v \in V : e_i v = v\}$. Also for i,j = 1, 2,..., t let $N_{ij} = e_i N e_j$. Then $N = \sum N_{ij}$. For i = 1,..., t let $B_i = \{w_i \in V_i : w_i = n_{ij} v_j$ for some $j \neq i, n_{ij} \in N_{ij}, v_j \in V_j\}$, and let W denote the subgroup of V generated by $B_1 \cup B_2 \cup ... \cup B_t$. Finally let $W_L = \{w \in V : f(w+v) = f(w) + f(v)$ for all $v \in V, f \in C(R;V)\}$. Therefore it is clear that $End_R(V) = C(R;V)$.

Corollary 2.1. If R is not semisimple then at least one A_i must be a ring.

Proof: Since $C(R; V)$ is semisimple then $R = S_1 \oplus ... \oplus S_k + N$ where N = *rad R* and each S_i is simple with identity e_i . As before let $N_{ij} = e_i N e_j$ and let W be the R-submodule of V as in Lemma1. If W = (*0*) then $N_{ij} V_j = (0)$ for each $i \neq j$ where V_j is the 1-space for e_j. This means each V_i is an R-module as well as $C(R; V)$-invariant. Hence

$$C(R;V) \cong C(R_1;V_1) \oplus ... \oplus C(R_k;V_k)$$

where $R_i = S_i + N_{ii}$. Since $C(R; V)$ is semisimple each $C(R_i;V_i)$ is semisimple [8,p. 146]. We show now that if $N_{ii} \neq (0)$ then $C(R_i;V_i)$ cannot be semisimple.

Suppose $N_{ii}{}^{l} = (0)$ but $N_{ii}{}^{l-1} \neq (0)$. Let $W = \ker N_{ii}{}^{l-1} = \{v \in V_i : nv = 0 \textit{ for all } n \in N_{ii}{}^{l-1}\}$, a proper subgroup of V_i , an S_i -submodule, and $C(R_i;V_i)$ -invariant. Asa $S_i -$ module V_i is completely reducible so $V_i = W_1 \oplus W_2$, an S_i -module direct sum. As constructed in the proof of Lemma 1 of [6] there exists a nonzero $f \in C(R_i;V_i)$ such that $f(V_i) \subseteq W_1$ and $f(W_1) = \{0\}$. Let $I = \{f \in C(R_i;V_i) : f(V_i) \subseteq W_1 \textit{ and } f(W_1) = \{0\}\}$. Then I is a nilpotent

$C(R_i;V_i)$-subgroup $(I^2=(0))$ and hence $C(R_i;V_i)$ is not semisimple. So each $N_{ii}=(0)$ and since $N_{ij}V=(0), N_{ij}=(0)$ *if* $i\neq j$. Thus *R* is semisimple.

So we may assume $W\neq(0)$. Since *W* is *C*(*R; V*)-invariant the map $f\to f/W$ is a homomorphism of *C*(*R; V*) into the ring $End_R(W)$. Hence a non-trivial homomorphic image of *C*(*R; V*) is a ring and this implies at least one simple component of *C*(*R; V*) is a ring [8, p. 55].

3. Simple Centralizer Near-ring

In this section we will discuss simple centralizer near-ring and ring with their characteristics or properties.

Theorem 3.1. *C*(*R; V*) is a ring when $End_R(V)\neq C(R;V)$.

Proof: We will prove this theorem with the help of an example. Let *R* is the ring consisting of the 3 x 3 matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},\quad a,b,c\in GF(2).$$

Let

$$V=\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}: x,y,z\in GF(2).\right.$$

A calculation shows that $End_R(V)=R$. Another calculation gives $f(Rv)\subseteq Rv$ for each $f\in C(R;V)$ and for each $v\in V$. From this it follows that *C*(*R; V*) is a ring since if $v\in V$ then

$$f(g+h)v=f(gv+hv)=f(r_1v+r_2v)=(r_1+r_2)f(v)=(fg+fh)v.$$

Let $\{e_1,e_2,e_3\}$ be the standard basis for the vector space *V* over *GF*(*2*). Then $V=R(e_1+e_2+e_3)\cup \mathrm{Re}_2\cup \mathrm{Re}_3$ and the relation $f(e_1+e_2+e_3)=f(e_2)=f(e_3)=e_1$ determines a function in *C*(*R; V*). But *f* is not in $End_R(V)$ since $f(e_2+e_3)\neq f(e_2)+f(e_3)$. Hence $End_R(V)\neq C(R;V)$.

Lemma 3.1. [2]. *W* is an *R*-submodule of *V*, W_L is a subgroup of *V* and $W\subseteq W_L$, where $W_L=\{w\in V: f(w+v)=f(w)+f(v)$ for all $v\in V, f\in C(R;V)\}$.

Proof: An element of *W* has the form $w=\sum n_{ij}v_j$ with $i\neq j$. For $n_{kl}\in N_{kl}$ and $n_{ij}v_j\in B_j$ we have $n_{kl}n_{ij}v_j\in B_k$ if $k\neq j$ and $n_{kl}n_{ij}v_j=n_{kl}(n_{ij}v_j)\in B_k$ if $k=j$. In this manner it is seen that $NW \subseteq W$. Also if $s \in S_1 \oplus ... \oplus S_t$ then $sn_{ij}v_j = (sn_{ij})v_j \in B_i$ since $sn_{ij} \in N_{ij}$. Hence $SW \subseteq W$ and W is an R-submodule of V.

The second part of the lemma is straight forward and is not discussed. To prove the last part it suffices to show that $B_i \subseteq W_L$ for each i. To this end let $v_i = n_{ij}v_j \in B_i$, $f \in C(R;V)$. For $k \neq i$ we have $f(v_i + v_k) = f(v_i) + f(v_k)$. For $v_i' \in V_i$,

$$f(v_i + v_i') = f(n_{ij}v_i + v_i') = f((n_{ij} + e_j)(v_j + v_i'))$$

$$= (n_{ij} + e_j) f(v_j + v_i') = (n_{ij} + e_j)[f(v_j) + f(v_i)] = f(v_i) + f(v_i').$$

With this it is easy to see that $f(v_i + v) = f(v_i) + f(v)$ for all $v \in V$, as desired.

Lemma 3.2. $C(R; V)$ would not be simple but it is a ring.

Proof: From Lemma 3.1, every $f \in C(R;V)$ is linear on W and moreover $f(W) \subseteq W$. Suppose now that $C(R; V)$ is simple. Then the map $f \to f/W$ is an imbedding of $C(R; V)$ into $End_R(W)$. Also $W \neq (0)$, for otherwise $N_{ij}V_j = (0)$ for each $i \neq j$ and so each V_i is an R-module and $C(R; V)$-invariant. Hence $C(R; V)$ would not be simple. Thus $W \neq 0$ and $C(R; V)$ is a ring.

4. Conclusion

Starting with the definition of centralizer near-ring throughout the paper we have discussed semisimple and simple centralizer near-rings with their various characteristics or properties. We have established Proposition 2.2, Theorem 2.1, Theorem 2.2, Corollary 2.1, Theorem 3.1 and Lemma 3.2.

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