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GENERALIZED DERIVATIONS ACTING AS HOMOMORPHISMS AND ANTI-HOMOMORPHISMS ON LIE IDEALS OF PRIME RINGS

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ABSTRACT

Let *U* be a non-zero square closed Lie ideal of a 2-torsion free prime ring *R* and *f* a generalized derivation of *R* with the associated derivation *d* of *R*. If *f* acts as a homomorphism and as an anti-homomorphism on *U*, then we prove that $d = 0$ or $U \subset Z(R)$, the centre of *R*.

Keywords: Prime ring, Lie ideal, Generalized derivation

1. Introduction

Let us consider *R* to be an associative ring with centre *Z*(*R*) throughout the article.

A ring *R* is said to be 2-torsion free if $2x = 0$ with $x \in R$, then $x = 0$. A ring *R* is called a prime ring if for any $x, y \in R$, $xRy = 0$ implies $x = 0$ or $y = 0$.

In a ring *R*, the symbol $[x, y]$ is known as the commutator of *x* and *y*, which is defined by $[x, y]$ = $xy - yx$, where $x, y \in R$. Two useful basic commutator identities are:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z .$$

An additive subgroup *U* of *R* is said to be a Lie ideal of *R* if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal *U* of *R* is called a square closed Lie ideal if $u^2 \in U$ for all $u \in U$.

An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + yd(x)$ for all $x, y \in R$. An additive mapping $f : R \to R$ is called a generalized derivation if there is a derivation $d : R \to R$ such that $f(xy) = f(x)y + yd(x)$ holds for all $x, y \in R$.

Let *S* be a non-empty subset of *R* and *f* a generalized derivation of *R*. If $f(xy) = f(x)f(y)$ [resp. $f(xy) = f(y)f(x)$] for all $x, y \in S$, then *f* is said to act as a homomorphism [resp. as an anti-homomorphism] on *S*.

The notion of generalized derivation was introduced by Bresar [4] and several characterizations of generalized derivation were obtained by B. Hvala [6] and T. K. Lee [7]. In [2], Bell and Kappe prove that if a derivation acts as a homomorphism and as an anti-homomorphism on a non-zero ideal I of a prime ring R, then $d = 0$. Asma, Rehman and Shakir [1] extend this result to a square closed Lie ideal, whereas Rehman [8] proves the same result for generalized derivations.

In this article, we extend the main result of [8] to square closed Lie ideals by using the similar arguments to get the following result.

Theorem 1.1 *Let $U \neq 0$ be a square closed Lie ideal of a 2-torsion free prime ring R, and f a generalized derivation of R with the associated derivation d of R.*

(i) *If f acts as a homomorphism on U, then $d = 0$ or $U \subset Z(R)$.*

(ii) *If f acts as an anti-homomorphism on U, then $d = 0$ or $U \subset Z(R)$.*

2. Main Results

We begin with the following two lemmas (established earlier) which are needed to accomplish the desired proof of our Theorem 1.1.

Lemma 2.1 ([3], Lemma 4) Let $U \not\subset Z(R)$ be a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = 0$. Then $a = 0$ or $b = 0$.

Lemma 2.2 ([3], Lemma 5) Let $U \neq 0$ be a Lie ideal of a 2-torsion free prime ring R and $d \neq 0$ a derivation of R such that $d(U) = 0$. Then $U \subset Z(R)$.

The following useful result plays an important role to reach the goal.

Lemma 2.3 If $U \neq 0$ is a Lie ideal of a 2-torsion free prime ring R such that $[U,U] = 0$, then $U \subset Z(R)$.

Proof. For all $u \in U$ and $x \in R$, we have

$$[u,[u,x]] = 0 \tag{1}$$

Replacing x by xy with $y \in R$, and using (1), we obtain

$$\begin{aligned} 0 &= [u, x[u, y] + [u, x]y] \\ &= x[u,[u, y]] + [u, x][u, y] + [u, x][u, y] + [u,[u, x]]y \\ &= 2[u, x][u, y]. \end{aligned}$$

Since R is 2-torsion free, we get

$$[u, x][u, y] = 0 \tag{2}$$

for all $u \in U$ and $x, y \in R$.

Putting yz for y in (2) with $z \in R$, and using (2), we obtain

$$[u, x]y[u, z] = 0 \text{ for all } u \in U \text{ and } x, y, z \in R.$$

Thus, we have $[u,x]R[u,z]=0$.

So, $[u,x]=0$ or $[u,z]=0$ for all $u\in U$ and $x,z\in R$ (by the primeness of R).

In both the cases, we see that $U\subset Z(R)$. □

We are now in a position to prove our main result in the following way.

Proof of Theorem 1.1 Let us assume that $U\not\subset Z(R)$.

Since U is a square closed Lie ideal, we have

$$uv+vu=(u+v)(u+v)-u^2-v^2\in U \text{ for all } u,v\in U.$$

Also, we get $uv-vu\in U$ for all $u,v\in U$.

So, $2uv\in U$ for all $u,v\in U$.

Therefore, $4(uvw)=2(2uv)w\in U$ for all $u,v,w\in U$.

(i) If f acts as a homomorphism on U, then we obtain

$$\begin{aligned} f(4uvw) &= f(2(2uv)w)=4f(uv)w+4uvd(w) \\ &= 4(f(u)f(v)w+uvd(w)) \end{aligned} \tag{3}$$

for all $u,v,w\in U$.

On the other hand,

$$\begin{aligned} f(4uvw) &= f(2u(2vw))=4f(u)f(vw) \\ &= 4(f(u)f(v)w+f(u)vd(w)) \end{aligned} \tag{4}$$

for all $u,v,w\in U$.

Comparing (3) and (4), and using the 2-torsion freeness of R, we get

$$f(u)vd(w)=uvd(w),$$

which yields

$$(f(u)-u)vd(w)=0 \tag{5}$$

for all $u,v,w\in U$.

Thus, we have $(f(u)-u)Ud(w)=0$ for all $u,w\in U$.

In view of Lemma 2.1, we obtain that

$$f(u)-u=0 \text{ for all } u\in U \text{ or } d(w)=0 \text{ for all } w\in U.$$

If $d(w)=0$ for all $w\in U$, then by Lemma 2.2, we have $d=0$ or $U\subset Z(R)$.

Since $U\not\subset Z(R)$, we get $d=0$.

On the other hand, if $f(u)-u=0$ for all $u\in U$, then we have

$$f(u)=u \tag{6}$$

for all $u\in U$.

Replacing *u* by 2*uv* in (6) for $v\in U$, and using the 2-torsion freeness of *R*, we get

$$uv=f(uv)=f(u)v+ud(v)=uv+ud(v) \text{ for all } u,v\in U.$$

Thus, we have $ud(v)=0$ for all $u,v\in U$.

Therefore, $Ud(v)=0$ for all $v\in U$.

Since $[U,R]\subset U$, we obtain $[U,R]d(v)=0$ for all $v\in U$.

This yields, $URd(v)=0$ for all $v\in U$.

Because $U\neq 0$ and *R* is prime, we have $d(v)=0$ for all $v\in U$.

Thus, by Lemma 2.1, we get $d=0$ or $U\subset Z(R)$. The fact $U\not\subset Z(R)$ forces $d=0$.

(ii) Let us suppose that *f* acts as an anti-homomorphism on *U*. Then we have

$$f(u)v+ud(v)=f(v)f(u)=f(uv) \tag{7}$$

for all $u,v\in U$.

Putting 2*uv* in place of *u* in (7), and using (7), we obtain

$$uvd(v)=f(v)ud(v) \tag{8}$$

for all $u,v\in U$.

Substituting 2*wu* for *u* in (8), we get

$$wuvd(v)=f(v)wud(v) \tag{9}$$

for all $u,v,w\in U$.

Multiplying (8) by *w* on the left, we have

$$wuvd(v)=wf(v)ud(v) \tag{10}$$

for all $u,v,w\in U$.

Comparing (9) and (10), we obtain

$$[w,f(v)]ud(v)=0 \tag{11}$$

for all $u,v,w\in U$.

In view of Lemma 2.1, we get

$$[w,f(v)]=0 \text{ for all } v,w\in U \text{ or } d(v)=0 \text{ for all } v\in U.$$

If $d(v)=0$ for all $v\in U$, then by Lemma 2.2, we find $d=0$ or $U\subset Z(R)$, and hence $d=0$, since $U\not\subset Z(R)$.

On the other hand, if

$$[w, f(v)] = 0 \tag{12}$$

for all $v, w \in U$, then upon replacing *v* by 2*vw* in (12), and using (12), we have

$$v[w, d(w)] + [w, v]d(w) = 0 \tag{13}$$

for all $v, w \in U$.

Again, substituting 2*uv* for *v* in (13) for $v \in U$, and using 2-torsion freeness of *R*, we get

$$\begin{aligned} 0 &= uv[w, d(w)] + [w, uv]d(w) \\ &= uv[w, d(w)] + u[w, v]d(w) + [w, u]vd(w) \\ &= u(v[w, d(w)] + [w, v]d(w)) + [w, u]vd(w) . \\ &= [w, u]vd(w) \text{, by using (13).} \end{aligned}$$

Thus, we obtain

$$[w, u]Ud(w) = 0 \tag{14}$$

for all $u, w \in U$.

Applying Lemma 2.1 in (14), we find that

$$d(w) = 0 \text{ for all } w \in U \text{ or } [w, u] = 0 \text{ for all } u, w \in U .$$

If $[w, u] = 0$ for all $u, w \in U$, the in view of Lemma 2.3, it follows that $U \subset Z(R)$, which is a contradiction to the fact that $U \not\subset Z(R)$. So, we have $d(w) = 0$ for all $w \in U$.

By using Lemma 2.2, we have $d = 0$ or $U \subset Z(R)$.

Since $U \not\subset Z(R)$, we conclude that $d = 0$. □

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