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JORDAN DERIVATIONS ON LIE IDEALS OF σ-PRIME RINGS

Akhil Chandra Paul[1] **and Md. Mizanor Rahman**[2,*]
[1] Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh.
[2] Department of Mathematics, Jagannath University, Dhaka-1100, Bangladesh.
[*]Corresponding author: mizanorrahman@gmail.com

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ABSTRACT

In this paper we prove that, if U is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and $d: R \to R$ is an additive mapping satisfying $d(u^2) = d(u)u + ud(u)$, for all $u \in U$ then $d(uv) = d(u)v + u\,d(v)$ holds for all $u, v \in U$.

Keywords: Lie ideal, σ-square closed Lie ideal, σ-prime ring, Jordan derivation, derivation.

1. Introduction

Throughout the paper, we consider R to be an associative ring with centre Z. $[a, b] = ab - ba$ which denotes the commutator of a and b, we will use the identities: $[ab, c] = [a, c]b + a[b, c]$ and $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in R$. An additive subgroup U of R is called a Lie ideal if $[U, R] \subseteq U$. An additive mapping $d: R \to R$ is called a derivation if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$ and it is called a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Clearly every derivation is a Jordan derivation but the converse is not true in general. A ring R is said to be a prime ring if $aRb = 0$ $(a, b \in R)$ implies that $a = 0$ or $b = 0$. An additive mapping $f: R \to R$ is called a generalized derivation with the associated derivation $d: R \to R$ if $f(ab) = f(a)b + a\,d(b)$ holds for all $a, b \in R$; it is called a Jordan generalized derivation with the associated derivation d of R such that $f(a^2) = f(a)a + ad(a)$ holds for all $a \in R$. R. Awtar [1] proved that if $U \not\subseteq Z$ is a square closed Lie ideal of a 2-torsion free prime ring R and $d: R \to R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u)$, for all $u \in U$ then $d(uv) = d(u)v + u\,d(v)$ holds for all $u, v \in U$.

We need the following lemmas due to R. Awtar [1] for proving our result.

Lemma 1.1 If $U \not\subseteq Z$ is a Lie ideal of a ring R, then $d(uv + vu) = d(u)v + ud(v) + d(v)u + vd(u)$ holds for all $u, v \in U$.

Lemma 1.2 If $U \not\subseteq Z$ is a Lie ideal of a ring R, then $d(uvu) = d(u)vu + ud(v)u + uvd(u)$ holds for all $u, v \in U$.

Lemma 1.3 If $U \not\subseteq Z$ is a Lie ideal of a ring R, then $d(uvw + wvu) = d(u)vw + ud(v)w + uvd(w) + d(w)\,vu + wd(v)u + wvd(u)$ holds for all $u, v, w \in U$.

Lemma 1.4 If $U \nsubseteq Z$ is a Lie ideal of a ring R, then $u^v[u,v]=0$ holds for all $u,v\in U$, where $u^v=d(uv)-d(u)\,v-u\,d(v)$.

Lemma 1.5 If $U \nsubseteq Z$ is a Lie ideal of a ring R, then $[u,v]\,u^v = 0$ for all $u,v\in U$, where u^v is as in Lemma1.4

2. Jordan Derivations on Lie Ideals of σ-Prime Rings

Let R be a ring. A mapping $\sigma: R\to R$ is called an involution if $\sigma(a+b)=\sigma(a)+\sigma(b), \sigma^2(a)=a$ and $\sigma(ab)=\sigma(b)\,\sigma(a)$ holds for all $a,b\in R$. A Lie ideal U of R is called a σ-Lie ideal if $\sigma(U)=U$ and it is called a σ-square closed Lie ideal if it is a σ-Lie ideal and for all $u\in U, u^2\in U.$ A ring R with involution σ is said to be a σ-prime ring if $aRb=aR\sigma(b)=\{0\}$ implies that a=0 or b=0. It is worthwhile to note that every prime ring having an involution σ is σ-prime but the converse is not true in general. As an example, let $T=R\times R^0$, where R^0 is an opposite ring of a prime ring R with involution $(x,y)=(y,x)$. Then T is not prime if $(0,a)T(a,0)=0$. But, R is σ-prime if we set $(a,b)T(x,y)=0$ and $(a,b)T\ \sigma((x,y))=0$, then $aRx\times yRb=0$ and $aRy\times xRb=0$ and thus $aRx=yRb=aRy=xRb=0$ by Oukhtite and Salhi [6]. We define the set $S_{a\sigma}(R)=\{x\in R: \sigma(x)=\pm x\}$ which are known as the set of symmetric and skew symmetric elements of R. Let U be a Lie ideal of R. We define $C_R(U)=\{r\in R: ru=ur, \forall\, u\in U\}$ which we shall call the centralizer of U with respect to R. Oukhtite and Salhi [12] worked on left derivation on σ-prime rings and proved that $U\subseteq Z$ or $d(U)=0$, where U is a nonzero σ-square closed Lie ideal of R. Oukhtite and Salhi [12] described additive mappings $d: R\to R$ such that $d(u^2)=2ud(u)\ \forall u\in U$, where U is a nonzero σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and prove that $d(uv)=ud(v)+vd(u)$ for all $u,v\in U$. Afterwords, Oukhtite, Salhi and Taoufiq [11] studied Jordan generalized derivations on σ-prime rings and proved that every Jordan generalized derivation on U of R is a generalized derivation on U of R, where U is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R. Some significant results on Lie ideals and generalized derivations in σ-prime rings have been obtained by M. S. Khan and M. A. Khan [5]. On the other hand, various remarkable characterizations of σ-prime rings on σ-square closed Lie ideals have been studied by many authors viz. M. R. Khan, D. Arora and M. A. Khan [4] ; Oukhtite and Salhi [7, 8, 9,10] and J. Bergun, I. N. Herstein and J. W. Kerr [2] and I. N. Herstein [3]. In this paper, we shall prove that if $d:R\to R$ is an additive mapping satisfying $d(u^2)=2ud(u)\ \forall u\in U$, where U is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R then $d(uv)=ud(v)+vd(u)$ for all $u,v\in U$ and hence every Jordan derivations on a σ-prime ring R is a derivation on R. We begin with the following results.

Lemma 2.1 Let R be a 2-torsion free σ-prime ring and U be a σ-Lie ideal of R. Let $u\in U$ be any element such that $[u,[u,x]]=0$, for all $x\in R$, then $[u,x]=0$.

Proof: We have $[u,[u,x]]=0$ for all $x\in R$. Let $y\in R$, then $xy\in R$. Replacing x by xy, we have $[u,[u,xy]]=0$. So $0=[u,[u,xy]]=[u,x[u,y]+[u,x]y]$

$$= [u, x[u, y]] + [u, [u, x]y]$$
$$= x[u, [u, y]] + [u, x][u, y] + [u, x][u, y] + [u, [u, x]]y$$
$$= 2[u, x][u, y].$$

Since R is 2-torsion free so $[u, x][u, y] = 0$. For every $z \in R$ we have $zy \in R$. Putting zx for y, we have $[u, x][u, zx] = 0$. Therefore, $0 = [u, x](z[u, x] + [u, z]x)$

$$= [u, x]z[u, x] + [u, x][u, z]x$$
$$= [u, x]z[u, x].$$

Therefore, $[u, x]R[u, x] = 0$. Since $\sigma(U) = U$, we have $\sigma(u) = u$, for all $u \in U$. Let $x \in S_{a\sigma}(R)$. Then $\sigma(x) = \pm x$. If $\sigma(u) = u$ and $\sigma(x) = -x$, then

$$\sigma([u, x]) = \sigma(ux - xu) = \sigma(ux) - \sigma(xu) = \sigma(x)\sigma(u) - \sigma(u)\sigma(x) = - xu + ux = [u, x].$$

Hence $[u, x]R[u, x] = [u, x]R\sigma[u, x] = 0$. By the σ-primeness of R, we get $[u, x] = 0$.

Lemma 2.2 Let R be a 2-torsion free σ-prime ring and $U \neq 0$ be a σ-Lie ideal and a σ- subring of R. Then either $U \subseteq Z$ or U contains a nonzero σ-ideal of R.

Proof: First we assume that, U as a σ-ring is not commutative. Then for some $u, v \in U$, $[u, v] \neq 0$ and $[u, v] \in U$. Therefore the ideal J of R generated by $[u, v]$ is nonzero, $J \subseteq U$ and $\sigma(J) = J$. On the other hand, let us assume that U is commutative. Then for every $u \in U$, $[u, [u, x]] = 0$ for all $x \in R$. Hence by Lemma 2.1, $[u, x] = 0$. This shows that $U \subseteq Z$.

Lemma 2.3 If $U \nsubseteq Z$ is a σ-Lie ideal of a σ-prime ring R, then $C_R(U) = Z$.

Proof: $C_R(U)$ is both a σ-subring and a σ-Lie ideal of R and $C_R(U)$ contains no nonzero σ-ideal of R. In view of Lemma 2.2, $C_R(U) \subseteq Z$. Therefore, $C_R(U) = Z$.

Lemma 2.4 If U is a σ-Lie ideal of a σ-prime ring R and $a \in R$. If $[a, [U, U]] = 0$ then $[a, U] = 0$, that is, $C_R([U, U]) = C_R(U)$.

Proof: If $[U, U] \nsubseteq Z$, then by Lemma 2.3, $a \in Z$, so a centralizes U. On the other hand, let $[U, U] \subseteq Z$, then we have $[u, [u, x]] = 0$ for $u \in U$ and $x \in R$. In view of Lemma 2.1, $[u, x] = 0$. This yields that $U \subseteq Z$. For both the cases we have seen that $a \in C_R(U)$. This gives that $C_R([U, U]) = C_R(U)$.

Lemma 2.5 Let $U \nsubseteq Z$ be a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and $d: R \rightarrow R$ be an additive mapping satisfying $d(u^2) = d(u)u + u\,d(u)$, for all $u \in U$. If $u^v = d(uv) - d(u)v - u\,d(v)$, for all $u, v \in U$ then $u^v w[u, v] = 0$, for all $w \in U$.

Proof: In view of Lemmas 1.4 and 1.5, we have $[u^v, [u, v]] = u^v[u, v] - [u, v]u^v = 0$. This yields that $u^v \in C_R([U, U]) = C_R(U)$, by Lemma 2.4. Hence for every $w \in U$, we have $u^v w[u, v] = 0$.

Lemma 2.6 ([7], Lemma 2.2) Let $U \nsubseteq Z$ be a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $aUb = aU\sigma(b) = 0$, then $a = 0$ *or* $b = 0$.

Theorem 2.7 Let U be a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and $d: R \rightarrow R$ be an additive mapping satisfying $d(u^2) = d(u)u + u\,d(u)$, for all $u \in U$, then $d(uv) = d(u)v + ud(v)$ holds for all $u, v \in U$.

Proof: If U is a non-commutative Lie ideal of R then $U \nsubseteq Z$. By Lemma 2.5, we have $a^b w[a, b] = 0$ for all $a, b, w \in U$. Let us assume that $a, b \in U \cap S_{a\sigma}(R)$. Since $\sigma(U) = (U)$, we have $\sigma[a, b] = [a, b]$ as $[a, b] \in U$. If $\sigma(b) = -b$ and $\sigma(a) = a$, then $\sigma([a, b]) = \sigma(ab - ba) = \sigma(b)\sigma(a) - \sigma(a)\sigma(b = -ba + ab = [a, b]$. Also, if $\sigma(b) = b$ and $\sigma(a) = -a$, then $\sigma[a, b] = [a, b]$. Therefore, we have $a^b w[a, b] = a^b w\, \sigma[a, b] = 0$. By applying the Lemma 2.6 in the above relation, we obtain that $a^b = 0$ or $[a, b] = 0$ for all $a, b \in U \cap S_{a\sigma}(R)$. Let $I_a = \{b \in U: a^b = 0\}$ and $j_a = \{b \in U: [a, b] = 0\}$. Then I_a and J_a are additive subgroups of U such that $I_a \cup J_a = U$. Then by Brauer's trick $I_a = U$ or $J_a = U$. Using the similar argument, we have $U = \{a \in U: U = I_a\}$ or $U = \{a \in U: U = J_a\}$. If $U = \{a \in U: U = J_a\}$ then $[a, b] = 0$, which yields that $U \subseteq Z$, by Lemma 2.2. Which is a contradiction to the fact that $U \nsubseteq Z$. So we have $U = \{a \in U: U = I_a\}$ and hence $a^b = 0$ for all $a, b \in U \cap S_{a\sigma}(R)$. This implies

$$d(ab) = d(a)b + ad(b), \forall\, a, b \in U \cap S_{a\sigma}(R) \dots\dots\dots \quad (1)$$

Now let $u, v \in U$. If we define $u_1 = u + \sigma(u)$, $u_2 = u - \sigma(u), v_1 = v + \sigma(v), v_2 = v - \sigma(v)$. Then $u_1, u_2, v_1, v_2 \in U \cap S_{a\sigma}(R)$ and we have $2u = u_1 + u_2, 2v = v_1 + v_2$. Therefore, in view of (1), we obtain

$$d(2u2v) = d(u_1 v_1 + u_1 v_2 + u_2 v_1 + u_2 v_2)$$

$$= d(u_1)v_1 + u_1 d(v_1) + d(u_1)v_2 + u_1 d(v_2) + d(u_2)v_1 + u_2 d(v_1) + d(u_2)v_2 + u_2 d(v_2)$$

$$= \big(d(u_1) + d(u_2)\big)v_1 + (u_1 + u_2)d(v_1) + \big(d(u_1) + d(u_2)\big)v_2 + (u_1 + u_2)d(v_2)$$

$$= d(u_1 + u_2)v_1 + 2ud(v_1) + d(u_1 + u_2)v_2 + 2ud(v_2)$$

$$= d(2u)v_1 + 2u\,d(v_1) + d(2u)v_2 + 2ud(v_2)$$

$$= 2d(u)(v_1 + v_2) + 2u\,d(v_1 + v_2)$$

$$= 2d(u)2v + 2u\,d(2v)$$

$$= 4d(u)v + 4u\,d(v).$$

Thus $4d(uv) = 4(d(u)v + u\,d(v))$. Since R is 2-torsion free, we obtain

$d(uv) = d(u)v + u\,d(v)$. If U is a commutative σ-Lie ideal of R, then by Lemma 2.2, $U \subseteq Z$. Therefore, by using 2-torsion freeness of R and in view of the Lemma 1.1, we have

$d(uv) = d(u)v + ud(v)$.

In view of above theorem, we obtain the following corollary.

Corollary 2.8 Let *R* be a 2-torsion free σ-prime ring. Then every Jordan derivations on *R* is a derivation on *R*.

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