গণিতপরিক্রমা ২৬শ খণ্ড ২০১৬ পৃ. ৪৫-৫৩

DISCUSSION ON SOME TOPICS OF TRIGONOMETRY : A SHORT AND SIMPLE PROOF OF THE "COT RELATIONS" AND THEIR USES

By
A. R. Khalifa*
Department of Mathematics, Dacca University

1. Authors of text-books on Statics and Mechanics admit in their books that the two "Cot Relations", variously known as two "Important Trigonometrical Theorems" or 'Results', prove very useful in simplification of many problems on Equilibrium of Forces. The proofs of the results are given in many of the text-books on Statics, and even in some treatises on Trigonometry. The relations have not obtained sufficient attention they deserve, in trigonometry.

The proof of the 'relations' found in these treatises is based on the Sine Rule established in the chapter on Properties of Triangles, discussed at a much later stage in trigonometry. The proof is so much complicated that it was set in Calcutta University Honours Examination, 1936. A very short, simple, direct and interesting proof is furnished below. It is based on the very definition of the cotangent ratio alone that is taught on the very first day, a student practically begins his trigonometry with these ratios. Further the usual proof requires both the parts to be separately treated in the same manner, whereas in the present proof one part is actually proved and the result is applied on the triangle itself to prove the second part; thus one part is something like the 'dual' of the other.

2. The "Cot Relations or Theorems or Formulae" are stated and proved below :

Statement : H is a point on the base RK of the triangle ARK, dividing the base in the ratio $m : n$, and the vertical angle RAK into two parts α and β, $\angle AHK = \theta$;

to prove that $(m + n) \cot\theta = n \cot R - m \cot K$; (1)

$$= m \cot \alpha - n \cot\beta. \quad (2)$$

* মৃত্যুবরণ: ঢাকা ডিসেম্বর ১৯৮৪ ।

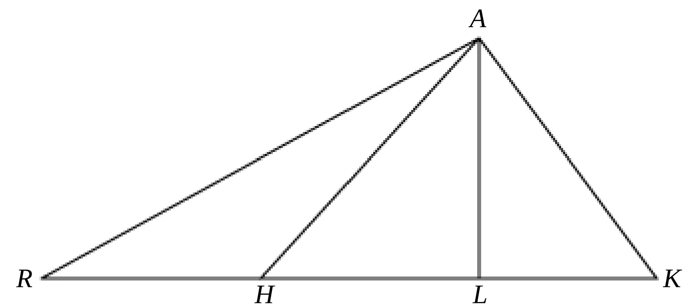
Construction. From A draw AL perpendicular to the base RK.

Proof. Since $RH : HK = m : n$, we may take $RH = md$, and $HK = nd$.

Now, $(m + n) \cot \theta = (m + n) \cot \angle AHK$,

$$= (m+n)\frac{HL}{AL} = \frac{n.HL + m.HL}{AL}$$

$$= \frac{n.HL - (-m.HL)}{AL};$$



this step is mere suggestive of the next step and is better omitted.

$$\frac{n.(md + HL) - m.(nd - HL)}{AL} = \frac{n.(RH + HL) - m.(HK - HL)}{AL}$$

$$= \frac{n, RL - m.LK}{AL} = n.\frac{RL}{AL} - m.\frac{LK}{AL}$$

$$= n \cot R - m \cot K.$$

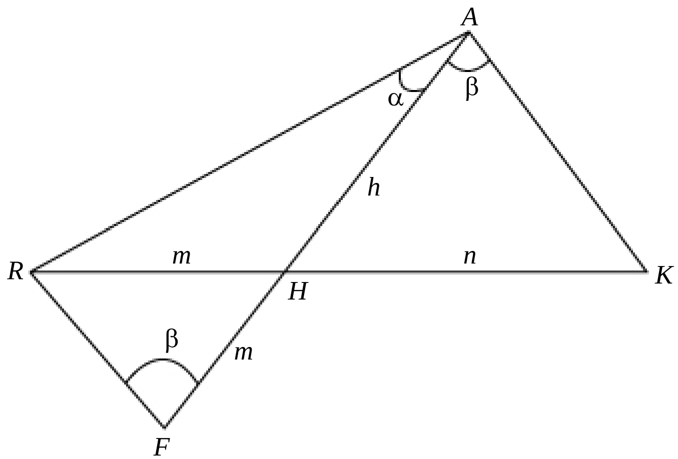
To prove the second part, through R draw RF parallel to AK to meet AH produced at F, so that the triangles RHF and KHA are similar,

$\therefore$ $HF : HA = RH : KH = m : n$; and the $\angle RFA =$ the alternate $\angle HAK = \beta$.

Now from the 'cot relation' (proved above) applied on the triangle RAF, RH dividing the base AF in the ratio $m : n$;

$(m + n) \cot \angle RHF = m \cot \angle RAF - n \cot \angle RFA$,

i.e. $(m + n) \cot\theta = m \cot\alpha - n \cot\beta$.



3. Nobody has been able to detect that the formula

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},$$

which is sometimes known as the ‘Tangent Rule’ (Napier’s Analogy in Plane Trigonometry), and which is so widely used in the solution of triangles, when two sides and their opposite angles are given, is but a simple deduction of the ‘‘Cot Relations.”

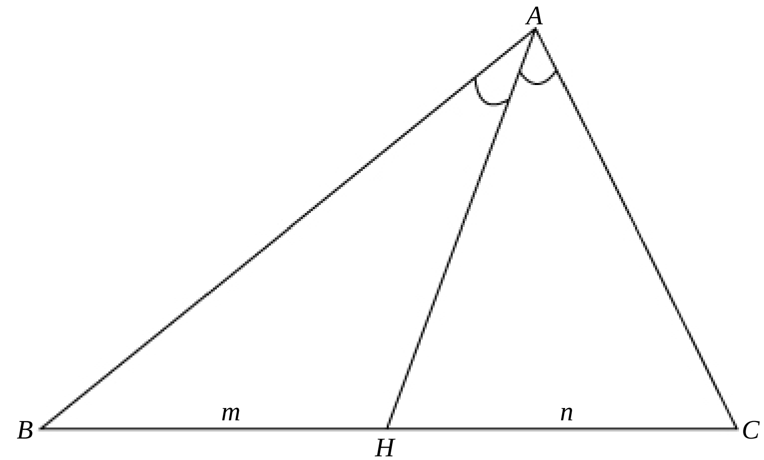
In the special case when the line *AH* bisects the vertical angle *A* of a triangle *ABC*, we have *BH* : *HC* = *AB* : *AC* = *c* : *b*,

and $\angle BAH = \angle CAH = \frac{A}{2}$

and $\angle\theta = \angle AHC = \angle BAH + \angle ABH = \frac{A}{2}+B,$

$$=\frac{A}{2}+\frac{B}{2}+\frac{C}{2}-\left(\frac{C}{2}-\frac{B}{2}\right),\ =\frac{\pi}{2}-\left(\frac{C}{2}-\frac{B}{2}\right);$$

so that here $\cot\theta = \tan\left(\frac{C}{2}-\frac{B}{2}\right),$



Then from the second relation of the “cot formula,”

$(c + b)\cot\theta = c\cot\angle BAH - b\cot\angle CAH;$

i.e. $-(b+c)\tan\left(\frac{B}{2}-\frac{C}{2}\right)=(c-b)\cot\frac{A}{2},$

i.e. $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$

“Cot Relations” are theorems of trigonometry, but it is very unfortunate these are scarcely used in trigonometry; they may be used in problems on ‘‘Heights & distances” in many examples.

4. We come across the following 4 classical examples (series) in the chapter on Summation of trigonometrical series:

1. (a) $\operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 2^2x + \ldots;$

   (b) $\operatorname{cosec} x + \operatorname{cosec}\frac{1}{2}x + \operatorname{cosec}\frac{1}{2^2}x + \ldots;$

2. (a) $\tan x + 2\tan 2x + 2^2\tan 2^2x + 2^3\tan 2^3x + \ldots;$

   (b) $\tan x + \frac{1}{2}\tan\frac{1}{2}x + \frac{1}{2^2}\tan\frac{1}{2^2}x + \frac{1}{2^3}\tan\frac{1}{2^3}x + \ldots;$

It is usual with the text-books to work out one or the other of these four and display the rest amongst the examples for exercise with hints under them. It is very difficult to remember and use these hints which serve as the generating relation or formula (if we may say so) for the terms of the several series. I had difficulty over 13 long years, till at last one day I found that the four came from the same source, and a very known source. So I call them cousins; the first two constituting the junior cousins and the last two the senior ones. I call them cousin series as they come from near the same stock.

The 4 trigonometrical expressions

$$\frac{1 \pm \cos x}{\sin x} \text{ and } \frac{1 \pm \sin x}{\cos x}$$

together with their 4 reciprocals are very important trigonometrical fractions which occur so frequently in differential calculus in connection with pedal equations, curvature, tangents and normals and the like; they are easily simplified

into tan or cot of an angle, in a non-fractional form. We take $\frac{1+\cos 2x}{\sin 2x}$ as the simple representative of all the 8 associates.

Simplifying, $\frac{1+\cos 2x}{\sin 2x} = \frac{2\cos^2 x}{2\sin x \cos x}$;

i.e. cosec $2x$ + cot $2x$ = cot x ; (A)

this is the generating formula for the two junior cousins. Now omit 1 from the numerator of the source fraction, and simplify as follows:

$\frac{\cos 2x}{\sin 2x} = \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x}$; *i.e.* cot $2x$ = $\frac{1}{2}$ (cot x – tan x),

i.e. tan x = cot x – 2 cot $2x$; (R)

this is the generating formula for the senior cousins.

These formulae (A) an (R) can be easily adjusted with the general term t_r; thereafter putting r = 1, 2, 3,........, n, n equalities are obtained, which, when added vertically, terms cancel from the right hand side, leaving the required sum for n terms, as the difference of

cotangents of two angles and in the case of senior cousins multiples of the cotangents.

The series may be given in a variety of forms to test the intelligence and ingenuity of gifted students, and a last term may be given which may not be the nth term, as for example the third series may be set in the form:

$$\tan \frac{1}{2}x + 2\tan x + 2^2 \tan 2x + 2^3 \tan 2^2 x + \ldots + 2^{n+1} \tan 2^n x.$$

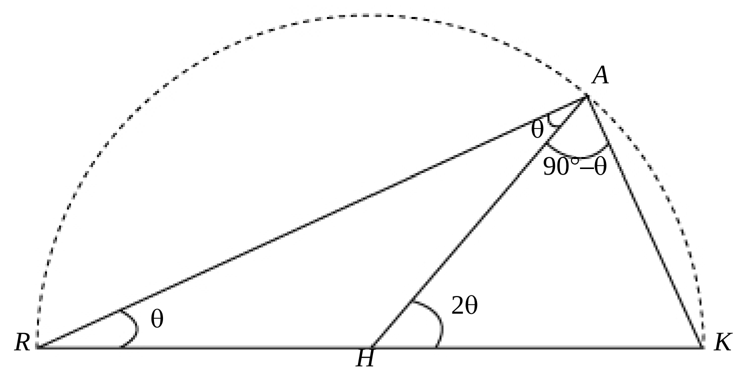
In all cases the relation or formula can be adjusted to the general term without difficulty.

5. After I set to work on the ‘‘cot formulae” I found that formula (R) is an easy application of these. When in the original triangle *ARK* the vertical angle *A* is a right angle, and *H* is the middle point of the base (now also hypotenuse) *RK*, the circle on *RK* as diameter passes through *A*,

so the *HR* = *HA*; taking ∠*RAH* as θ, ∠*HAK* = $\frac{\pi}{2}$ – θ, ∠*ARH* = ∠*RAH* = θ, exterior ∠*AHK* = 2θ; *m* : *n* = 1 : 1, so that from the “cot formula” we get (1 +1) cot ∠*AHK* = cot ∠*RAH* – cot ∠*HAK*,

i.e. 2 cot 2θ = cotθ – cot $\left(\frac{\pi}{2} - \theta\right)$ = cotθ – tanθ;

i.e. tanθ = cotθ – 2 cot 2θ ; this is (R).



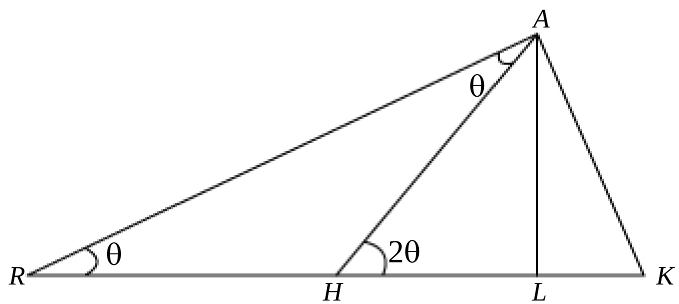
Now it will not be difficult to know and remember the relation (R) for the 2 senior cousin series.

The formula (A) does not follow from the “Cot formula”, but can be easily obtained from the above triangle, by drawing perpendicular *AL* from *A* to *RK*.

Now cosec $\angle AHL = \frac{AH}{AL} = \frac{RH}{AL} = \frac{RL - HL}{AL} = \frac{RL}{AL} - \frac{HL}{AL}$

$= \cot \angle R - \cot \angle AHL,$

i.e. cosec 2θ = cot θ – cot 2θ ; which is (A).



6. Recently I had an opportunity to look into available text-books on trigonometry, both elementary and of a higher level, originating from U.S.A., Great Britain, Bharat, and Pakistan, and I find to my utter surprise that there is much room for improvement in dealing with the problems; many of the examples can be worked out very easily by alternative better methods and use of proper technique, in a much shorter time. I take two examples, one from higher, and the other from elementary trigonometry

Ex. 1. Prove that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$

$$= \cos\left(n\frac{\pi}{2} - n\theta\right) + i\sin\left(n\frac{\pi}{2} - n\theta\right)$$

$$= (\sin\theta + i\cos\theta)^n.$$

This sum has been worked out in many books in a very round-about manner. It can be worked as follows:

We know, $\sin^2\theta + \cos^2\theta = 1$, so that

$$\frac{\sin\theta + i\cos\theta}{1} = \frac{1}{\sin\theta - i\cos\theta} = \frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}$$

by $\frac{N_1}{D_1} = \frac{N_2}{D_2} = \frac{N_1+N_2}{N_1+D_2}$;

so that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = (\sin\theta + i\cos\theta)^n$

$$= \left\{\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^n$$

$$= \cos\left(n\frac{\pi}{2} - n\theta\right) + i\sin\left(n\frac{\pi}{2} - n\theta\right).$$

Ex. 2. Prove that $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\sin\theta + 1}{\cos\theta}$

(Ex. 17 worked out in page 72 of Schaum’s Outline of Theory and Problems of Plane and Spherical Trigonometry, New York, 1954, by Frank Ayres, Jr., Ph. D.)

This has been worked out in 6 very lengthy steps. This may be worked out as follows:

$\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta - 1 = -\cos^2\theta$,

so that $\frac{\sin\theta+1}{\cos\theta} = \frac{-\cos\theta}{\sin\theta-1} = \frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$;

by $\frac{N_1}{D_1} = \frac{N_2}{D_2} = \frac{N_1+N_2}{D_1+D_2}$

Thus we have shown: right hand side = left hand side. It may not be out of place to exhibit here an artificial solution of an example in trigonometry.

Prove that tan 70° = 2 tan 50° + tan 20.°

(This is worked-out Ex. 5 page 89 of Intermediate Trigonometry by Md. Iman Ali, B.A. (Hons.), M.A.; K.N.S. Gold Medalist).

It has been solved as follows :

"L.S. = tan (90° – 20° = cot 20°.

R.S. = 2 tan (90° – 40°) + tan 20° = 2 cot 40 + tan 20°

$$=2\cot(20° + 20°) + \tan 20° = \frac{2(\cot^2 20°-1)}{2\cot 20°} + \tan 20°$$

$$= \frac{\cot^2 20°-1}{\cot 20°}+\frac{1}{\cot 20°}=\frac{\cot^2 20°}{\cot 20°} = \cot 20°."$$

But the problem is very simply worked out as follows, without using the formula for cot 2*A* or cot (*A* + *B*), which nobody cares to remember:

tan 50° = tan (70° – 20°)

$$= \frac{\tan 70°-\tan 20°}{1+\tan 70° \tan 20°}=\frac{\tan 70°-\tan 20°}{2},$$

since tan 70° tan 20° = tan 70° cot 70° = 1;

$$\therefore\ 2\tan 50° = \tan 70° - \tan 20°$$

whence, tan 70° = 2 tan 50° + tan 20°.

A. R. Khalifa (আজিজর রহমান খলিফা : ১৯০৪-৮৪)। স্বনামখ্যাত গণিতবিদ। কোলকাতা বিশ্ববিদ্যালয়ের ১৯২৭-এর গণিতে এম.এ.। দীর্ঘদিন (১৯৪৮-৬৭, ১৯৭৩-৭৫) ঢাকা বিশ্ববিদ্যালয়ে অধ্যাপনা করেছেন। নিবেদিতপ্রাণ একজন শিক্ষক হিসেবে প্রশংসিত হয়েছেন। 'গণিত কাঁদিয়া ফেরে' তার একটি অপ্রকাশিত মূল্যবান রচনা। *The Dacca University Studies*, Vol. XII, June 1964 Pt. B, pp. 57-63 প্রকাশিত তার এই পেপার খুবই গুরুত্ব বহন করে বিধায় অনেক পাঠকের একান্ড় অনুরোধে পরিক্রমার জন্য তা পুনমুর্দ্রণ করা হলো। - **m¤úv`K**

W