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TL-IDEALS OF TL-SUBNEAR-RINGS

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ABSTRACT

The aim of this paper is to introduce and study TL-ideals of TL-subnear-rings. Also we define T-sum, T-difference and T-product of L-subsets of a near-ring R and obtain their properties.

Key words: Fuzzy set, TL-subnear-ring, TL-ideal, Homomorphism, Near-ring

1. Introduction

Near-Ring is a generalized structure of a ring. The theory of fuzzy sets was introduced by Zadeh [15] .The fuzzy set theory has been developed in many directions by the research scholars. Goguen [9] introduced the concept of L-fuzzy sets. Rosenfeld [13] first introduced the fuzzification of the algebraic structures and defined fuzzy subgroups. Anthony and Sherwood [3], Asaad and Abou-zaid [4], Akgul [2], Das [6], Dixit, Bhambri and Kumar [7] contributed the theory of fuzzy subgroups. Fuzzy ideals of rings are first defined by Liu. [11] and the study was continued by many other researchers to extend the concepts.

Abou-Zaid [1] introduced the notion of fuzzy R-subgroups and fuzzy ideals of near-rings. Dutta and Biswas [8] introduced fuzzy and fuzzy cosets of fuzzy ideals of near-rings. Cheng, Mordeson and Yandong [5] have discussed TL-subnear-rings and TL-ideals of a ring.

As in ring theory, it is interesting to fuzzify some substructures of near-ring. Hence our aim in this paper is to study TL-ideals of TL-subnear-rings and to characterize them.

2. Preliminaries

We recall some definitions for the sake of completeness.

Definition (2.1) [10]: By a near-ring we mean a non-empty set R with two binary operations '+' and '·' satisfying the following axioms:

(i) (R, +) is a group,

(ii) (R, ·) is a semi-group,

(iii) $x\cdot(y + z) = x\cdot y + x\cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word "near-ring" instead of "left near-ring". We denote xy instead of $x\cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$, but $0x \neq 0$ for $x, y \in R$.

Definition (2.2) [1, 8]: An ideal I of a near-ring R is a subset of R such that

(i) (I, +) is a normal subgroup of (R, +),

(ii) $RI \subseteq I$,

(iii) $(r + i)s - rs \in I$ for all $i \in I$ and $r, s \in R$.

Note that if I satisfies (i) and (ii) then it is called a left ideal of R.

If I satisfies (i) and (iii) then it is called a right ideal of R.

Definition (2.3): A binary operation T on a lattice L is called a t-norm if it satisfies the following conditions:

(i) $T(T(a, b), c) = T(a, T(b, c))$,

(ii) $T(a, b) = T(b, a)$,

(ii) $b \leq c \Rightarrow T(a,b) \leq T(a,c)$,

(iv) $T(a, 1) = a$,

for all $a, b, c \in L$.

Definition (2.4): A fuzzy set μ in a near-ring R is a function $\mu: R \to [0, 1]$.

Definition (2.5): Let μ be a fuzzy set in a near-ring R and $t \in [0, 1]$.Then the crisp set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called a t-level subset or t-cut of μ.

3. TL-ideals of TL-subnear-ring

Let R be a near-ring and L be a complete lattice.

Definition (3.1)[14]: An L-subset μ of a near-ring R is called a TL-subgroup of R if it satisfies the following conditions:

(i) $\mu(0) = 1$,

(ii) $\mu(-x) \geq \mu(x)$,

(iii) $\mu(x+y) \geq \mu(x)\, T\, \mu(y)$.

for all $x,y \in R$.

Definition (3.2)[14]: An L-subset μ of a near-ring R is called a TL- subnear-ring of R if it satisfies the following conditions:

(i) $\mu(0) = 1$,

(ii) $\mu(-x) \geq \mu(x)$,

(iii) $\mu(x+y) \geq \mu(x) \mathrm{T} \mu(y)$,

(iv) $\mu(xy) \geq \mu(x) \mathrm{T} \mu(y)$ for all $x, y \in R$.

Remarks (3.3): (i) When $T = \wedge$, a TL-subnear-ring is called L-subnear-ring.

(ii)The set of all TL- subnear-rings of R and set of all L- subnear-rings of R are denoted by TL(R) and L(R) respectively.

(iii) If L = [0, 1], TL- subnear-ring and L- subnear-ring of R are known as T-fuzzy subnear-ring and fuzzy subnear-ring of R respectively.

Definition (3.4)[14]: An L-subset μ of a near-ring R is called a TL-ideal of R if

(i) $\mu(0) = 1$,

(ii) $\mu(-x) \geq \mu(x)$,

(iii) $\mu(x + y) \geq \mu(x) \mathrm{T} \mu(y)$,

(iv) $\mu(y + x - y) \geq \mu(x)$,

(v) $\mu(xy) \geq \mu(y)$,

(vi) $\mu((x + i)y - xy) \geq \mu(i)$ for

for all $x, y, i \in R$.

Remarks (3.5) : (i) If μ satisfies (i), (ii), (iii), (iv) and (v) then it is TL-left ideal of R and if μ satisfies (i), (ii), (iii), (iv) and (vi) then it is TL-right ideal of R.

(ii) When $T = \wedge$, a TL-left ideal and TL-right ideal are known as L-left ideal and L-right ideal respectively.

(iii) The set of all TL- left ideals and TL-right ideals of R are denoted by $TLI_l(R)$ and $TLI_r(R)$ respectively.

(iv) When $T = \wedge$, set of all L- left ideals and L-right ideals of R are denoted by $LI_l(R)$ and $LI_r(R)$ respectively.

(v) When L = [0, 1] TL-left ideals and TL-right ideals are known as T-fuzzy left ideals and T-fuzzy right ideals of R respectively and when $T = \wedge$, they are known as fuzzy left ideals and fuzzy right ideals of R respectively.

Now we define TL-ideals of TL-subnear-rings:

Definition (3.6): A function $\mu: R \rightarrow L$ is called an L-subset of R.

The set of all L-subsets of R is called the L-power set of R and is denoted by L^R.

Definition (3.7): Let $\mu \in L^R$, $\gamma \in TL(R)$ and $\mu \leq \gamma$. Then

(1) μ is a TL-left ideal of γ if

 (i) μ is a normal TL-subgroup of (R,+),

 (ii) $\mu(xy) \geq \gamma(x) T \mu(y)$ for all x, y $\in$R.

(2) μ is a TL-right ideal of γ if

 (i) μ is a normal TL-subgroup of (R,+),

 (ii) $\mu((x+i)y - xy) \geq \mu(i) \, T \, \gamma(y)$ for all x, y, i $\in$R.

(3) μ is said to be TL-two sided or TL-ideal of γ if

 (i) μ is a normal TL-subgroup of (R,+),

 (ii) μ is both TL-left and TL-right ideal of γ.

Remark (3.8): When T= $\wedge$ these ideals will be called as L-left, L-right and L-two sided ideals of γ respectively.

Theorem (3.9): An L-subset $\mu \in L^R$ is a TL-right (resp.TL-left) ideal of R if and only if μ is a TL-right (resp.TL-left) ideal of the TL-subnear-ring 1_R.

Proof: Part (I): Let $\mu \in L^R$ be a TL-right ideal of R.

Then $\mu((x+i)y - xy) \geq \mu(i) \leq$ for all x, y, i $\in$R.

Now $\mu(i) \, T \, 1_R(y) \leq \mu(i) \wedge 1_R(y) = \mu(i)$.

$\Rightarrow \mu(i) \, T \, 1_R(y) \leq \mu(i) \leq \mu((x+i)y - xy)$.

i.e. $\mu((x+i)y - xy) \geq \mu(i) \, T \, 1_R(y)$ for all x, y, i$\in$R.

Hence μ is a TL-right ideal of the TL-subnear-ring 1_R.

Conversely let μ be a TL-right ideal of the TL-subnear-ring 1_R.

Therefore $\mu((x+i)y - xy) \geq \mu(i) \, T \, 1_R(y)$ for all x, y, i $\in$R.

$\Rightarrow \mu((x+i)y - xy) \geq \mu(i) \, T \, 1_R(y) = \mu(i)$.

$\Rightarrow \mu((x+i)y - xy) \geq \mu(i)$ for all x, y, i $\in$R.

Hence μ is a TL-right ideal of R.

Part (II): Let $\mu \in L^R$ be a TL-left ideal of R.

Then μ is a normal TL-subgroup of (R, +) and $\mu(xy) \geq \mu(y)$.

Therefore $1_R(x) \, T \, \mu(y) \leq 1_R(x) \wedge \mu(y) = \mu(y)$ for all x,y $\in$R.

$\Rightarrow 1_R(x) \, T \, \mu(y) \leq \mu(y) \leq \mu(xy)$.

Hence μ is a TL-left ideal of the TL-subnear-ring 1_R.

Conversely let μ be a TL-left ideal of the TL-subnear-ring 1_R.

Therefore $\mu (xy) \geq 1_R (x)\ T\ \mu (y) = \mu (y)$.

Hence μ is a TL-left ideal of R. ■

Theorem (3.10): Let $\mu \in L^R$ and $\gamma \in L(R)$.If μ is L- right (resp.L-left) ideal of γ then for every $a \in L$, μ_a is a right (resp. left) ideal of γ_a.

Proof: Part (I): Let μ be L-right ideal of γ .

Then $\mu ((x+i) y - xy) \geq \mu (i)\ T\gamma (y)$.

Let $x \in \mu_a$ and $y \in \gamma_a$.

Then $x \in \mu_a$ and $y \in \gamma_a$. $\Rightarrow \mu (x), \gamma (y) \geq a$.

Therefore $\mu (y+x-y) = \mu(x) \geq a$ for all $x \in \mu_a$ and $y \in \gamma_a$.

$\Rightarrow y+x-y \in \mu_a$ for all $x \in \mu_a$ and $y \in \gamma_a$.

Hence μ_a is a normal L-subgroup of $(\gamma_a,+)$.

Now let $i \in \mu_a$ and $x,y \in \gamma_a$

Then $\mu(i) \geq a$ and $\gamma (x) \geq a$, $\gamma (y) \geq a$.

since $\mu((x+i)y - xy) \geq \mu(i) \wedge \gamma (y) \geq a$.

Therefore $(x+i)\ y - xy \in \mu_a$ for all $i \in \mu_a$ and $x, y \in \gamma_a$.

Hence μ_a is a right ideal of γ_a.

Part (II): Let μ be L- left ideal of γ .

Let $x \in \mu_a$ and $y \in \gamma_a$.

Then $x \in \mu_a$ and $y \in \gamma_a \Rightarrow \mu(x), \gamma (y) \geq a$.

Then $\mu(y+x-y) \geq \mu(x) \geq a$ for all $x \in \mu_a$ and $y \in \gamma_a$.

Therefore $y+x-y \in \mu_a$ for all $x \in \mu_a$ and $y \in \gamma_a$.

Hence $(\mu_a ,+)$ is a normal L-subgroup of $(\gamma_a,+)$.

Now let $y \in \mu_a$ and $x \in \gamma_a$

Since $\mu (xy) \geq \gamma (x) \wedge \mu (y)$.

Therefore $\mu (xy) \geq \gamma (x) \wedge \mu (y) \geq a$.

Thus $xy \in \mu_a$ for all $y \in \mu_a$ and $x \in \gamma_a$.

Hence μ_a is a left ideal of γ_a. ■

Remark (3.11): The converse of the theorem is true only when $\mu \in L^R$ is a normal L-subgroup of (R, +).

Theorem (3.12): Let $\mu \in L^R, \gamma \in L(R)$ and L be a chain. Then a necessary condition for μ to be a L-right (resp.L-left) ideal of γ is that for every $a\in L\backslash\{1\}$, $\mu_{[a]}$ is a right (resp. left) ideal of $\gamma_{[a]}$.

Proof: Part (I): Let μ be a L-right ideal of γ.Then

(i) $\mu(0) = 1$ implies that $0\in \mu_{[a]}$ for every $a\in L\backslash\{1\}$.

(ii) $\mu(-x) \geq \mu(x)$ for $x \in \mu_{[a]}$ implies that $-x \in \mu_{[a]}$.

(iii) Let $x,y \in \mu_{[a]}$ .Then $\mu(x) > a$, $\mu(y) > a$.

But L is a chain, therefore either $\mu(x) \geq \mu(y)$ or $\mu(y) \geq \mu(x)$.

Assume that $\mu(y) \geq \mu(x)$.

As $\mu(x+y) \geq \mu(x) \wedge \mu(y) = \mu(x) > a$. Therefore $x + y \in \mu_{[a]}$ for all $i \in x, y \in \mu_{[a]}$.

Hence $\mu_{[a]}$ is a L-subgroup of R.

(iv) If $\mu_{[a]}$ is not normal then for some $a\in L\backslash\{1\}$, there exists $y \in R$ and $x \in \mu_{[a]}$ such that $y+x-y \notin \mu_{[a]}$ .Thus $\mu(x) > a$ and $\mu(y+x-y) \leq a$.

Hence $\mu(y+x-y) < \mu(x)$ and hence μ is not normal which is a contradiction.

Thus $\mu_{[a]}$ is a normal subgroup of R for all $a\in L\backslash\{1\}$.

(v) Again if $\mu_{[a]}$ is not a right ideal of $\gamma_{[a]}$ then for some $a\in L\backslash\{1\}$, there exists $x,y \in R$ and $i \in \mu_{[a]}$ such that $(x+i)y - xy \notin \mu_{[a]}$

Thus $\mu(i) > a$ and $\mu((x+i)y - xy) \leq a$.

This implies that μ is not L-right ideal.

Thus we get a contradiction.

Hence $\mu((x+i)y - xy) > a$ for all $i \in \mu_{[a]}$ and $x, y\in\gamma_{[a]}$.

Hence $\mu_{[a]}$ is a right ideal of $\gamma_{[a]}$.

Part (II): Let μ be a L-left ideal of γ.

Then as in part (I), $(\mu_{[a]},+)$is a normal L-subgroup of (R,+).

Again if $\mu_{[a]}$ is not a left ideal of $\gamma_{[a]}$ then for some $a\in L\backslash\{1\}$, there exists $y \in \mu_{[a]}$ and $x\in\gamma_{[a]}$ such that $xy\notin\mu_{[a]}$.

Then $\mu(y) > a$ and $\mu(xy) \leq a$ .Thus $\mu(xy) < \mu(y)$

Therefore $xy\notin \mu_{[a]}$ for $y \in \mu_{[a]}$,$x\in\gamma_{[a]}$,which is a contradiction.

Hence $\mu_{[a]}$ is a right ideal of $\gamma_{[a]}$. ■

Theorem (3.13): Let $\mu\in L^R$ and $\gamma \in L(R)$ and L be dense. Then a sufficient condition for μ to be a L-right (resp.L-left) ideal of γ is that for every $a\in L\backslash\{1\}$, $\mu_{[a]}$ is a right (resp. left) ideal of $\gamma_{[a]}$.

Proof: Part (I): Let us suppose that for every $a\in L\backslash\{1\}$, $\mu_{[a]}$ is a right ideal of $\gamma_{[a]}$.

(i) Clearly $\mu(0) = 1$.

(ii) Take $a < \mu(x)$.Then $x \in \mu_{[a]}$ implies $-x \in \mu_{[a]}$.

$\Rightarrow \mu(-x) > a \Rightarrow \mu(-x) \geq 1$ and $\mu(x) > a$.

$\Rightarrow \mu(-x) \geq \mu(x)$ for all $x \in \mu_{[a]}$.

(iii) Now suppose $x, y \in \mu_{[a]}$

Let $\mu(x) \wedge \mu(y) > a$.

Therefore $\mu(x) > \mu(x) \wedge \mu(y)$.

$\Rightarrow \mu(x) > a \Rightarrow x \in \mu_{[a]}$.Similarly $y \in \mu_{[a]}$.

Hence $x+y \in \mu_{[a]}$ and so $\mu(x+y) > a$.

Let $a = \mu(x) \wedge \mu(y)$.

If $a = 0$ then $\mu(x+y) \geq 0 = \mu(x) \wedge \mu(y)$.

If $a > 0$ then for any $b \in L$, $b < a$ we observe that $\mu_{[b]}$ is a right ideal of R

and $x,y \in \mu_{[b]}$, implies that $\mu(x+y) \in \mu_{[b]}$.

i.e. $\mu(x+y) > b \Rightarrow \mu(x+y) \geq \vee\{b \mid b \in L, b < a\}$.

Since L is dense, $\vee\{b \mid b \in L, b < a\} = a$.

Therefore $\mu(x+y) \geq a = \mu(x) \wedge \mu(y)$ for all $x, y \in \mu_{[a]}$.

(iv) Now let $y \in \gamma_{[a]}$ and $x \in \mu_{[a]}$ and let $a > 0$.

Then for any $b \in L$,$b < a$ we observe that $(\mu_{[b]}, +)$ is a normal subgroup of $(R,+)$, $x \in \mu_{[b]}$, implies $\mu(y+x-y) > b$.

Thus $\mu(y+x-y) \geq \vee\{b \mid b \in L, b < a\}$.

Since L is dense, $\vee\{b \mid b \in L, b < a\} = a$.

Therefore $\mu(y+x-y) \geq \mu(x)$ for all $i \in \mu_{[a]}$, $y \in \gamma_{[a]}$.

(v) Finally, let $i \in \mu_{[a]}$ and $x, y \in \gamma_{[a]}$.

If $a = 0$ then clearly $\mu((x+i)y-xy) \geq \mu(i)$.

If $a > 0$ then for any $b \in L$, $b < a$ we observe that $\mu_{[b]}$ is a right ideal of $\gamma_{[a]}$ and $x \in \mu_{[b]}$, implies $\mu((x+i)y-xy) > b$.

Hence $\mu((x+i)y-xy) \geq \vee\{b \mid b \in L, b < a\}$.

Since L is dense, $\vee\{b \mid b \in L, b < a\} = a$.

Therefore μ ((x+i) y-xy) $\geq$ a = μ (i) $\wedge\gamma$ (y).

Hence μ ((x+i) y-xy) $\geq$ μ (i) $\wedge\gamma$(y) for all x, y, i $\in$R.

Thus μ is a L-right ideal of γ.

Part (II): Let us suppose that $\mu_{[a]}$ is left ideal of $\gamma_{[a]}$ for every a$\in$ L\{1}

We shall prove the last condition which conforms that μ is a L-left ideal of γ.

Let y$\in$ $\mu_{[a]}$ and x$\in\gamma_{[a]}$.Then μ (y) > a and γ(x) > a.

If a=0 then clearly μ (xy) $\geq \gamma$(x) $\wedge$ μ (y).

If a > 0 then for any b$\in$L, b < a we observe that $\mu_{[b]}$ is a left ideal of $\gamma_{[a]}$ and x $\in$ $\mu_{[b]}$ implies μ (xy) > b. Hence μ (xy) $\geq$ $\vee$ {b| b$\in$ L, b < a}.

Since L is dense, $\vee$ {b| b$\in$ L, b < a}= a.

Therefore μ (xy) $\geq$ a = γ(x) $\wedge$ μ (y).

Hence μ (xy) $\geq$ a = γ(x) $\wedge$ μ (y) for all x, y $\in$R.

Thus μ to be a L-left ideal of γ. ■

Theorem (3.14): Let $\gamma\in$ TL(R) and μ be a TL-left ideal of γ.

Then Rμ is a left ideal of Rγ.

Proof: Let $\gamma\in$ TL(R) and μ be a TL-left ideal of γ.

Rμ = {x$\in$R| μ(x) = 1}, similarly Rγ = {x$\in$R| γ (x) = 1}.

We know that $\mu \leq \gamma$ and a$\in$L $\Rightarrow$ $\mu_{[a]} \subseteq \gamma_{[a]}$. Therefore R$\mu \subseteq$ Rγ.

Now first we shall prove that (Rμ,+) is a subgroup of (R,+).

(i) Since μ(0) = 1, 0 $\in$ Rμ .

Hence Rμ is a non-empty subset of R.

(ii) Let x $\in$ Rμ.

Then μ(x) = 1 and μ (-x) $\geq$ μ(x) implies μ (-x) = 1.

Therefore -x $\in$ Rμ for all x $\in$ Rμ.

(iii) Let x, y $\in$ Rμ.

Then μ(x+y) $\geq$ μ(x) T μ(y) $\Rightarrow$ μ(x+y) $\geq$ 1T1 = 1.

Therefore x+y$\in$ Rμ for all x, y$\in$ Rμ.

Hence (Rμ,+) is a subgroup of (R,+).

(iv)Let y $\in$ R and x $\in$ Rμ.

Since $\mu(y+x-y) \geq \mu(x)$ and $\mu(x) = 1$, $\mu(y+x-y) = 1$.

Therefore $y+x-y \in R\mu$ for all $y \in R$ and $x \in R\mu$.

Hence $(R\mu, +)$ is a normal subgroup of $(R, +)$.

Similarly $(R\gamma, +)$ is also a normal subgroup of $(R, +)$.

Again since $R\mu \subseteq R\gamma$, $(R\mu, +)$ is a normal subgroup of $(R\gamma, +)$.

(v) Since μ is a TL-left ideal of γ , $\mu(xy) \geq \gamma(x) \, T \, \mu(y)$ for all $x, y \in R$.

Let $x \in R\mu$ and $r \in R\gamma$.

Then $\mu(x) = 1$ and $\gamma(r) = 1$.

Therefore $\mu(rx) \geq \gamma(r) \, T \, \mu(x) = 1$.

Hence $rx \in R\mu$ for all $r \in R\gamma$ and $x \in R\mu$.

i.e. $R\gamma \, R\mu \subseteq R\mu$.

Hence $R\mu$ is a left ideal of $R\gamma$. ■

Similarly we can obtain the following theorem:

Theorem (3.15): Let $\gamma \in TL(R)$ and μ be a TL-right ideal of γ.

Then $R\mu$ is a right ideal of $R\gamma$. ■

Theorem (3.16): Let $\mu \in TLI_l(R)$ and γ be a normal TL-subgroup of $(R, +)$. Then $\mu T \gamma$ is a TL-left ideal of γ.

Proof: Let $\mu \in TLI_l(R)$ and $\gamma \in TL(R)$.

Clearly $\mu T\gamma \leq \gamma$ and $\mu T\gamma$ is a TL-subgroup of $(R, +)$.

Again $\mu T\gamma\,(y+x-y) = \mu(y+x-y) \, T \, \gamma(y+x-y) \geq \mu(x) \, T \, \gamma(x) = \mu T\gamma(x)$.

Therefore $\mu T\gamma(y+x-y) \geq \mu T\gamma(x)$ for all $x, y \in R$.

Next $\mu T\gamma\,(xy) = \mu(xy) T\gamma(xy) \geq \mu(y) T\gamma(x) \, T\gamma(y) = \gamma(x) \, T(\mu T\gamma)(y)$ for all $x, y \in R$.

Hence $\mu T\gamma$ is a TL-left ideal of γ. ■

Similarly we can prove the following theorem:

Theorem (3.17): Let $\mu \in TLI_r(R)$ and γ be a normal TL-subgroup of $(R, +)$. Then $\mu T\gamma$ is a TL-left ideal of γ ■

Theorem (3.18): Let $\xi \in TL(R)$ and μ, γ be two TL-left ideals of ξ .Then $\mu \wedge \gamma$ is a TL-left ideal of ξ.

Proof: Let $\xi \in TL(R)$ and μ, γ be two TL-left ideals of ξ .

Since μ and γ are TL-left ideals of ξ, $\mu \leq \xi$, $\gamma \leq \xi$ and so $\mu \wedge \gamma \leq \xi$.

Hence $\mu \wedge \gamma$ is a TL-subgroup of (R, +).

Again $(\mu \wedge \gamma)(y+x-y) = \mu(y+x-y) \wedge \gamma(y+x-y) \geq \mu(x) \wedge \gamma(x) = (\mu \wedge \gamma)(x)$, $x \in R$.

$(\mu \wedge \gamma)(xy) = \mu(xy) \wedge \gamma(xy) \geq (\xi(x)\, T\, \mu(y)) \wedge (\xi(x)\, T \gamma(y))$.

$= \xi(x)\, T\, (\mu(y) \wedge \gamma(y)) = \xi(x)\, T\, (\mu \wedge \gamma)(y)$ for all $x,y \in R$.

Hence $\mu \wedge \gamma$ is a TL-left ideal of ξ. ■

Similarly we can prove the following theorem:

Theorem (3.19): Let $\xi \in TL(R)$ and μ,γ be two TL-right ideals of ξ .Then $\mu \wedge \gamma$ is a TL-right ideal of ξ. ■

4. Homomorphism

The following definitions are well-known:

Definition (4.1): Let R and R' be two near-rings. A function f: R→R' is called a homomorphism if for all $x, y \in R$

(i) $f(x+y) = f(x) + f(y)$,

(ii) $f(xy) = f(x) f(y)$.

We know that a one-one homomorphism is an isomorphism.

Definition (4.2) Extension Principle:

Let X and Y be two non-empty sets and $f: X \rightarrow Y$ be a function. Then f induces two functions,

$f : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ and $f^{-1}: \mathcal{F}(Y) \rightarrow \mathcal{F}(X)$ which are defined as follows:

1) $[f(\mu)](y) = \sup_{x|y=f(x)} \{\mu(x)\}$; if $y=f(x)$,

$= 0$; otherwise.

2) $[f^{-1}(\gamma)](x) = \gamma(f(x)$; for all $\gamma \in \mathcal{F}(Y)$.

In the following theorem we prove that the homomorphic image of TL-left (resp.right) ideal of R is a TL-left (resp.right) ideal of S:

Theorem (4.3): Let $f: R \rightarrow S$ be a homomorphism of a near-ring R onto a near-ring S and $\mu \in TLI_r(R)$(resp.$\mu \in TLI_l(R)$).Then $f(\mu) \in TLI_r(S)$)(resp.$\mu \in TLI_l(S)$).

Proof: Let $f: R \rightarrow S$ be a homomorphism of a near-ring R onto a near-ring S.

Let $x, y \in S$.

Part (I): Let $\mu \in TLI_r(R)$.

(i) Clearly $f(\mu)(0') = 1$.

(ii) $f(\mu)(-x) = \vee\{\mu(w)|w\in R, f(w) = -x\}$.

$= \vee\{\mu(-w)|-w\in R, f(-w) = x\}$.

$\geq \vee\{\mu(w)|w\in R, f(w) = x\}$.

$= f(\mu)(x)$ for all $x\in S$.

(ii) $f(\mu)(x-y) = \vee\{\mu(w)|w\in R, f(w) = x-y\}$

$\geq \vee\{\mu(u-v)|u, v\in R, f(u) = x, f(v) = y\}$

$\geq \vee\{\mu(u)T\mu(v)|u, v \in R, f(u) = x, f(v) = y\}$

$\geq (\vee\{\mu(u))|u\in R, f(u) = x\})T(\vee\{\mu(v)| v \in R, f(v) = y\}$

$= f(\mu)(x)T f(\mu)(y)$ for all $x, y \in S$.

(iii) $f(\mu)(y+x-y) = \vee\{\mu(w)|w \in R, f(w) = y + x - y\}$

$= \vee\{\mu(v + u - v) |u,v \in R, f(u) = x, f(v) = y\}$.

$\geq \vee\{\mu(u) |u \in R, f(u) = x\}$.

Therefore $f(\mu)(y + x - y) \geq \vee\{\mu(u)|u\in R, f(u) = x\}$.

i.e. $f(\mu)(y + x - y) \geq f(\mu)(x)$ for all $x, y\in S$.

Hence $f(\mu)(y+x-y) \geq f(\mu)(x)$ for all $x, y\in S$.

(v) $f(\mu)((x+i)y-xy) = \vee\{\mu(w)|w \in R, f(w) = (x + i)y - xy\}$.

$\geq \vee\{\mu((u + t)v - uv)| u, v, t \in R, f(u) = x, f(v) = y, f(t) = i\}$.

$\geq \vee\{\mu(t)| t \in R, f(t) = i\}$.

$= f(\mu)(i)$ for all $x, y, i \in S$.

Hence $f(\mu) \in TLI_r(S)$. ≥

Part (II): Let $\mu \in TLI_l(R)$.Then clearly

(vi) $f(\mu)(xy) \geq f(\mu)(y)$ for all $x, y \in S$.

Hence from (i), (ii), (iii), (iv) and (vi) $f(\mu) \in TLI_l(S)$. ■

In the following theorem we discuss about the inverse images of TL-right/left ideals of R:

Theorem (4.4): Let $f : R \to S$ be a homomorphism of a near-ring R into a near-ring S and $\gamma\in TLI_r(S)$(resp.$\gamma\in TLI_l(S)$).Then $f^{-1}(\gamma)\in TLI_r(R)$ (resp.$f^{-1}(\gamma)\in TLI_l(R)$.

Proof: Let f: R → S be a homomorphism of a near-ring R onto a near-ring S.

Part (I): Let $\gamma \in TLI_r(S)$.

(i) $f^{1}(\gamma)(0) = \gamma(f)(0) = \gamma(0') = 1$.

(ii) $f^{1}(\gamma)(-x) = \gamma(f)(-x) = \gamma(-f(x)) \geq \gamma(f(x)) = f^{-1}(\gamma)(x)$ for all $x \in R$.

(iii)$f^{1}(\gamma)(x+y) \geq (f^{-1}(\gamma)(x))T(f^{-1}(\gamma)(y))$ for all $x,y \in R$.

(iv)$f^{1}(\gamma)(y-x+y) = \gamma(f(y+x-y)) \geq \gamma(f(x)) = f^{-1}(\gamma)(x)$ for all $x, y \in R$.

(v)$f^{1}(\gamma)((x+i)y-xy) = \gamma(f((x+i)y-xy)) \geq f^{-1}\gamma(i)$ for all $x, y, i \in R$.

Hence $f^{1}(\gamma) \in TLI_r(R)$.

Part (II) : Let $\gamma \in TLI_l(S)$.

(vi)$f^{1}(\gamma)(xy) = \gamma(f(xy)) = \gamma(f(x) f(y)) \geq \gamma(f(y)) = f^{1}(\gamma)(y)$ for all $x, y \in R$.

Hence from (i), (ii), (iii), (iv) and (vi) $f^{1}(\gamma) \in TLI_l(R)$. ■

For $\mu \in L^R$ and $\gamma \in TL(R)$ then $f(\mu)$ is a TL-left ideal of $f(\gamma)$ under the following conditions:

Theorem (4.5): Let f : R → S be a homomorphism of a near-ring R onto a near-ring S. Let $\gamma \in TL(R)$ and μ a TL-left ideal of γ.Then $f(\mu)$ is a TL-left ideal of $f(\gamma)$.

Proof: Let f: R → S be a homomorphism of a near-ring R onto a near-ring S.

Since $\mu \leq \gamma$, $f(\mu) \leq f(\gamma)$.

Part (I): Let μ be a TL- right ideal of γ.

Also both $f(\mu)$ and $f(\gamma)$ are TL-subgroups of (R,+).

Let $x, y \in S$. Then

$$\begin{aligned} f(\mu)(y+x-y) &= \vee\{\mu(w) | w \in R, f(w) = y+x-y\}. \\ &\geq \vee\{\mu(v+u-v) | u, v \in R, f(u) = x, f(v) = y\}. \\ &= \vee\{\mu(u) | u \in R, f(u) = x\}. \\ &= f(\mu)(x) \text{ for all } x, y \in R. \end{aligned}$$

$$\begin{aligned} f(\mu)((x+i)y - xy) &= \vee\{\mu(w) | w \in R, f(w) = (x+i)y - xy\} \\ &\geq \vee\{\mu((u+t)v - uv) | u, v, t \in R, f((u+t)v - uv)) = (x+i)y - xy\} \\ &\geq \vee\{\mu(t) T \gamma(v) | t, y \in R, f(t) = i, f(y) = v\} \\ &= (\vee\{\mu(t)) | t \in R, f(t) = i\}) T (\vee\{\gamma(v) | y \in R, f(y) = v\}). \\ &= f(\mu)(i) \, T \, f(\gamma)(y). \end{aligned}$$

Thus $f(\mu)((x + i)y - xy) \geq f(\mu)(i)\ T\ f(\gamma)(y)$, for all $x, y, i \in R$.

Hence $f(\mu)$ is a TL-right ideal of $f(\gamma)$. ■

Part (II): Let μ be a TL- left ideal of γ.

$f(\mu)(xy) = \vee\{\mu(w) | w\in R, f(w) = xy\}$.

$\geq \vee\{\mu(uv) | u, v\in R, f(u) = x, f(v) = y\}$.

$\geq \vee\{\gamma(u) T \mu(v) | u, v\in R, f(u) = x, f(v) = y\}$.

$= (\vee\{\gamma(u) | u \in R, f(u) = x\}) T (\vee\{\mu(v) | v \in R, f(v) = y\})$.

$= f(\gamma)(x)\ T\ f(\mu)(y)$ for all $x, y\in R$.

Hence $f(\mu)$ is a TL-left ideal of $f(\gamma)$. ■

For $\mu\in L^R$ and $\gamma \in TL(R)$ then $f(\mu)$ is a TL-left ideal of $f(\gamma)$, this we prove in the following theorem:

Theorem (4.6): Let $f:R\rightarrow S$ be a homomorphism of a near-rings and $\gamma\in TL(S)$, μ be TL-right (resp.left) ideal of γ.Then $f^{-1}(\mu)$ is a TL-right(resp.left) ideal of $f^{-1}(\gamma)$.

Proof: Let $f: R \rightarrow S$ be a homomorphism of a near-rings.

Clearly $\mu \leq \gamma \Rightarrow f^{-1}(\mu) \leq f^{-1}(\gamma)$.

As proved earlier $f^{-1}(\mu)$ and $f^{-1}(\gamma)$ are TL-subgroups of R and $f^{-1}(\mu) \leq f^{-1}(\gamma)$.

Part (I): Let μ be a TL- right ideal of γ and $x, y\in R$. Then

(i) $f^{-1}(\mu)(0) = \mu(f)(0) = \gamma(0') = 1$.

(ii) $f^{-1}(\mu)(-x) = \mu(f)(-x) = \mu(-f(x)) \geq \mu(f(x)) = f^{-1}(\mu)(x)$ for all $x \in R$.

(iii) $f^{-1}(\mu)(x-y) = \mu(f)(x-y) \geq \mu(f(x)) T \mu(f(y)) = f^{-1}(\mu)(x)\ T\ f^{-1}(\mu)(y)$.

i.e. $f^{-1}(\mu)(x-y) \geq f^{-1}(\mu)(x)\ T\ f^{-1}(\mu)(y)$ for all $x,y \in R$.

(iv)$f^{-1}(\mu)(y+x-y) = \mu(f)(y+x-y) = \mu(f(y)+f(x)-f(y)) \geq \mu(f(x)) = f^{-1}(\mu)(x)$.

i.e. $f^{-1}(\mu)(y+x-y) \geq f^{-1}(\mu)(x)$ for all $x,y\in R$.

(v)$f^{-1}(\mu)((x+i)y-xy) = \mu(f)((x+i)y-xy)$.

$= \mu((f(x) + (f(i))(f(y)- (f(x)(f(y))$.

$\geq \mu(f(i))$.

$= f^{-1}(\mu)(i)$.

Thus $f^{-1}(\mu)((x+i)y-xy) \geq \mu(f(i)) = f^{-1}(\mu)(i)$ for all $x, y, i\in R$.

Hence $f^{-1}(\mu)$ is a right ideal of $f^{-1}(\gamma)$.

Part (II): Let μ be a TL- left ideal of γ.Then (vi)$f^{-1}(\mu)(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq \gamma(f(x)T\mu(f(y)) = (f^{-1}(\gamma)(x))T(f^{-1}(\mu)(y))$.

i.e. $f^{1}(\mu)(xy) \geq (f^{1}(\gamma)(x))\ T\ (f^{1}(\mu)(y))$ for all x, y $\in$R.

Hence from (i), (ii) (iii) (iv) and (vi) $f^{-1}(\mu)$ is a TL-left ideal of $f^{1}(\gamma)$. ■

Theorem (4.7): If an L-subset μ^* of R/μ is defined by $\mu^*(x+\mu) = \mu(x)$ for all x∈R then $\mu^* \in$TLI(R).

Proof: Now let us define a function f: R/μ → R/R μ by

$f(x+\mu) = x + R\mu$ for all x ∈R.

We shall prove that f is an onto isomorphism.

(1) $f((x+\mu)+_T(y+\mu)) = f((x+y)+\mu) = (x+y) + R\mu = (x+R\mu)+(y+R\mu)$

$= f(x+\mu) + f(y+\mu))$.

(2) $f((x+\mu)_*(y+\mu)) = f(xy+\mu) = xy + R\mu = (x+R\mu)_*(y+R\mu)$.

$= f(x+\mu)_*f(y+\mu)$.

(3) Let $f(x+\mu) = f(y+\mu)$ where x, y ∈R.

Then $f(x+\mu) = f(y+\mu) \Rightarrow x + R\mu = y + R\mu \Rightarrow x\text{-}y \in R\mu \Rightarrow \mu(x-y) = \mu(0)$

$\Rightarrow \mu(x) = \mu(y) \Rightarrow x = y \Rightarrow x+\mu = y+\mu$.

Hence f is an isomorphism.

(4) We observe that for all $x + R\mu \in R/R\mu$, x ∈R there exists $x + \mu \in R/\mu$ such that $f(x+\mu) = x + R\mu$ for all x ∈R.

Therefore f is an onto isomorphism.

(5) An L-subset μ^* of R/μ is defined by

$\mu^*(x+\mu) = \mu(x)$ for all x ∈ R. Then

(i) $\mu^*(\mu) = \mu^*(0+\mu) = \mu(0) = 1$.

(ii) $\mu^*(-x+\mu) = \mu(-x) \geq \mu(x) = \mu^*(x+\mu)$ for all x ∈ R.

(iii) $\mu^*((x+\mu) +_T (y+\mu)) = \mu^*((x+y)+\mu) = \mu(x+y)$.

$\geq \mu(x)T\mu(y) = \mu^*(x+\mu)T\mu^*(y+\mu)$ for all x, y ∈ R.

(iv) $\mu^*((y+\mu) +_T (x+\mu) +_T(-y+\mu)) = \mu^*((y+x)+\mu) +_T(-y+\mu))$.

$= \mu^*((y+x)+\mu) +_T(-y+\mu))$.

$= \mu^*((y+x-y)+\mu)$.

$= \mu(y+x-y)$

$\geq \mu(x)$.

$= \mu^*(x+ \mu)$ for all x, y$\in$R.

(iv)$\mu^*((x + \mu)_*(y + \mu)) = \mu^*(xy + \mu) = \mu(xy) \geq \mu(y) = \mu^*(y + \mu)$ for all x, y $\in$ R.

(v) $\mu^*[((x + \mu) +_T (a + \mu)))_*(y + \mu)- (x + \mu)_*(y + \mu)]$

$= \mu^*[(x + a)_*(y + \mu) - (x + \mu)_*(y + \mu)].$

$= \mu^*[(x + a)y + \mu)- (xy + \mu)].$

$= \mu^*[((x + a)y – xy) + \mu)].$

$= \mu ((x + a)y – xy).$

$\geq \mu (a).$

$= \mu^*(a+ \mu)$ for all x,y,a$\in$R.

Therefore μ^* is a TL-left as well TL-right ideal of R. ■

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