GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) **38** (2018) 119-125

REAL ANALYTICITY OF HAUSDORFF DIMENSION FUNCTION OF DISCONNECTED JULIA SETS OF PARABOLIC POLYNOMIALS $f_{a,b}(z) = z(1 - az - bz^2)$

Hasina Akter
Department of Mathematics & Physics, North South University, Dhaka, Bangladesh
Email: hasina.akter@northsouth.edu

Received**:** 27-07-2018 Accepted: 18-12-2018

ABSTRACT

In the paper [3],we have shown that the function $D_0 \ni \lambda \longmapsto \mathrm{HD}\big(J(f_\lambda)\big) \in \mathbb{R}$ is real-analytic, where D_0 is the set of all parameters λ.In this paper, using a holomorphic function $\kappa : (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C} \setminus \{0\}$, $\kappa\big((a,b)\big) = \frac{b}{a^2}$ and the conjugacy property, we show that the Hausdorff dimension function $\widetilde{D_0} \ni (a,b) \longmapsto \mathrm{HD}\left(J(f_{a,b})\right) \in \mathbb{R}$ is real analytic, where $\widetilde{D_0} = \kappa^{-1}(D_0)$. Thisfunctionascribestothepolynomial$f_{a,b}$theHausdorffdimension of its Julia set$J(f_{a,b})$.

Keywords: Holomorphic dynamics, Hausdorff dimension, Julia sets, Real analyticity.

1. Introduction

Hausdorff dimension is used to measure roughness of objects in mathematics. Hausdorff dimension as a function of subsets of a given metric space usually behaves extremely irregularly. For example, if $n \geq 1$ and $\mathrm{K}(\mathbb{R}^n)$ denotes the space of all non-empty compact subsets of the Euclidean space $\mathbb{R}^n$, then the function $\mathrm{K}(\mathbb{R}^n) \ni K \longmapsto HD(K) \in \mathbb{R}$, ascribing to the compact set K its Hausdorff dimension $HD(K)$, is discontinuous at every point. It is therefore surprising indeed that the function $c \longmapsto HD(J_c)$ is continuous, where c belongs to M_0, the main cardioid of the Mandelbrot set $\mathcal{M}$, and $J_c$denotes the Julia set of the polynomial $\mathbb{C} \ni z \longmapsto z^2 + c$. Because of it's irregular behavior, the real analyticity of the Hausdorff dimension function has been studied by many mathematicians for different types of dynamical systems. In the paper [3], we have proved that the Hausdorff dimension function $D_0 \ni \lambda \longmapsto \mathrm{HD}\big(J(f_\lambda)\big) \in \mathbb{R}$ is real-analytic for the family of cubic polynomials

$$f_\lambda(z) = z\,(1 - z - \lambda z^2),$$

by using the theory of parabolic and hyperbolic graph directed Markov systems with infinite number of edges.

In this paper, by using the conjugacy property, we show that the Hausdorff dimension function $\widetilde{D_0} \ni (a,b) \longmapsto \mathrm{HD}\left(J(f_{a,b})\right) \in \mathbb{R}$ is real analytic, where $\widetilde{D_0} = \kappa^{-1}(D_0)$ for a holomorphic function

$$\kappa : (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C} \setminus \{0\},\ \kappa\big((a,b)\big) = \frac{b}{a^2}.$$

We know that [2] "a holomorphic endomorphism $T : J(T) \to J(T)$ is expansive if and only if $J(T)$ contains no critical point of T and an expansive holomorphic endomorphism $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is not expanding if and only if T has at least one parabolic fixed (periodic) point. It has been proved already by Fatou that all parabolic fixed (periodic) points for $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are contained in $J(T)$. A rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called parabolic [2] if its restriction to the Julia set $J(T)$ is expansive but not expanding, equivalently, if the Julia set contains no critical points but contains at least one parabolic periodic point."

2. Review of the Family $f_\lambda(z) = z\,(1 - z - \lambda z^2)$

This section underscores the important detail of the family

$$f_\lambda(z) = z\,(1 - z - \lambda z^2),\ \boldsymbol{\lambda} \in D_0 \subsetneq \mathbb{C} \setminus \{0\}$$

from the paper [3], where D_0 is the set of parameters $\boldsymbol{\lambda} \in \mathbb{C} \setminus \{0\}$defined by

$$D_0 = \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \begin{array}{l} f_\lambda \text{ has no non} - \text{zero parabolic or finite attracting periodic points} \\ \text{and one finite critical point of } f_\lambda \text{ escapes to } \infty \end{array} \right\}.$$

We have seen in [3] that the set of parameters D_0contains a deleted neighborhood of 0, and for each $\lambda \in \mathbb{C} \setminus \{0\}$, we get

$$f_\lambda'(z) = 1 - 2z - 3\lambda z^2.$$

The finite fixed points of f_λ are the solutions to the equation $f_\lambda(z) = z$. We see that,

$$f_\lambda(0) = 0 \qquad \text{and } f_\lambda\left(-\frac{1}{\lambda}\right) = -\frac{1}{\lambda}$$

and we also have

$$f_\lambda'(0) = 1 \qquad \text{and} f_\lambda'\left(-\tfrac{1}{\lambda}\right) = 1 - \tfrac{1}{\lambda}.$$

Hence we get the followings from [3].

"The number 0 is a parabolic fixed point of f_λ with multiplicity equal to 2 and with one petal. The ray $[0, +\infty)$forms its attracting direction and the ray $(-\infty, 0]$ forms its repelling direction [4]. Since any two polynomials bi-Lipschitz conjugate on their Julia sets have the same moduli of multipliers at corresponding periodic points, so if $\lambda, \gamma \in \mathbb{C} \setminus \{0\}$ such that$\left|1 - \frac{1}{\lambda}\right| \neq \left|1 - \frac{1}{\gamma}\right|$, then f_λ and f_γ are not bi-Lipschitz conjugate on their Julia sets. In particular, if $\lambda, \gamma \in (0, 1)$ and $\lambda \neq \gamma$, then $f_\lambda$and f_γ are not bi-Lipschitz conjugate on their Julia sets. Moreover, if $g(z) = \bar{z}$, then $f_\lambda(z) = (g \circ f_{\bar{\lambda}} \circ g^{-1})(z)$, that is f_λ and $f_{\bar{\lambda}}$ are bi-Lipschitz conjugate and their dynamics are symmetric about the real axis. Also, they share same topological and geometrical properties."

We have also shown in the paper [3] that one of the two critical points of f_λ approaches ∞ and other one is contained in the parabolic basin of 0, and as a consequence, we get that $f_\lambda$is a parabolic polynomial, and for each $\boldsymbol{\lambda}\in D_0$ the Julia set $J(f_\lambda)$ is disconnected[4]. By using the theory of parabolic and hyperbolic graph directed Markov systems with infinite number of edges([5], [6]), we have proved in the paper [3] that the Hausdorff dimension function $D_0 \ni \lambda \longmapsto \mathrm{HD}\big(J(f_\lambda)\big) \in \mathbb{R}$ is real-analytic.

2.1 Revision of the set of parameters,D_0

Here we will give few more detail of the set D_0. In the paper [3], we have assumed that the set D_0 is open, but we have skipped the proof. Here we will show that the setD_0 is open and is contained in some left half-plane.

Lemma 2.1 The set of parameters D_0 is open.

Proof. To prove the Lemma, first we will prove that for all compact set $K \subset \mathbb{C}$, there exists a neighborhood $\widehat{U}$ of ∞ such that

$$\forall\ \lambda \in K,\ f_\lambda\big(\widehat{U}\big) \subseteq\ \widehat{U}.$$

Consider the function $h(z) = \frac{1}{z}$ and define $g_\lambda = h^{-1} \circ f_\lambda \circ h$. Then

$$g_\lambda(z) = \frac{1}{f_\lambda\left(\frac{1}{z}\right)} = \frac{z^3}{z^2 - z - \lambda} = z + \frac{z^2 + \lambda z}{z^2 - z - \lambda},$$

and hence

$$g'_\lambda(z) = 1 - \frac{(1+\lambda)z^2 + 2\lambda z + \lambda^2}{(z^2 - z - \lambda)^2}.$$

We have $g_\lambda(0) = 0$ and $g'_\lambda(0) = 0$. For some small $0 < \epsilon < 1$, define

$$\mathcal{U} = \big\{(\lambda, z) \in K \times \hat{\mathbb{C}} : |g'_\lambda(z)| < 1 - \epsilon\big\}.$$

Then $\mathcal{U} \neq \emptyset$ since $K \times \{0\} \subseteq \mathcal{U}$, that is, $\mathcal{U}$ is a non-empty open set. There exist $U, V \subset \mathbb{C}$ such that $K \subset U$, $\{0\} \subset V$ with $U \times V \subset \mathcal{U}$. Since V is an open set containing 0, without loss of generality we may assume that $V = B(0, \delta)$ for some $\delta > 0$. By Mean Value Inequality, for all $z \in V = B(0, \delta)$

$$|g_\lambda(z) - g_\lambda(0)| \leq (1 - \epsilon)|z - 0| \Rightarrow |g_\lambda(z)| < (1 - \epsilon)\delta < \delta$$

and so we have $g_\lambda(V) \subseteq V$ for all $\lambda \in K$.

Now h is an open map, implies $h(V)$is open and $h(0) = \infty \in h(V)$. Denote $\infty \in h(V) = \widehat{U}$, an open neighborhood of ∞. Since $g_\lambda = h^{-1} \circ f_\lambda \circ h$, for all $\lambda \in K$ we have

$$f_\lambda\big(\widehat{U}\big) = f_\lambda(h(V) = h\big(g_\lambda(V)\big) \subseteq h(V) = \widehat{U}.$$

Now we will prove the Lemma.

The finite critical points of f_λ are the solutions to the equation $f_\lambda'(z) = 0$, which are$c_\lambda = \frac{-1 \pm \sqrt{1+3\lambda}}{3\lambda}$.By definition of D_0, a finite critical point of $f_\lambda$must escape to ∞.Since $\lambda = -\frac{1}{3} \Rightarrow c_{-\frac{1}{3}} = 1$ is a double critical point and $\lim_{n\to\infty} f^n_{-\frac{1}{3}}(1) = 0$, so $\lambda = -\frac{1}{3} \notin D_0$. Fix $\lambda_0 \in D_0$. Consider the compact set $K = \bar{B}(\lambda_0, 1)$. Then there exists a neighborhood $\hat{U}$ of ∞ such that

$$\forall\ \boldsymbol{\lambda} \in K,\ f_\lambda(\hat{U}) \subseteq \hat{U}.$$

Since $\lambda_0 \in D_0$, there exists $c_{\lambda_0} \in \mathrm{Crit}(f_\lambda)$ and a positive integer $N = N(\lambda_0)$such that

$$f^n_{\lambda_0}(c_{\lambda_0}) \in \hat{U} \quad \forall\, n \geq N(\lambda_0).$$

Take a small neighborhood ofλ_0, say $B(\lambda_0, r)$, where r is small enough so that $-\frac{1}{3} \notin B(\lambda_0, r) \subseteq \bar{B}(\lambda_0, 1)$.

Define

$$X = \{(\lambda, c) : \lambda \in B(\lambda_0, r) \text{ and } c \in \mathrm{Crit}(f_\lambda)\},$$

and define a function

$$\rho : X \to B(\lambda_0, r) \quad by \quad \rho\big((\lambda, c)\big) = \lambda.$$

Then ρ is analytic and since the gradient of the function ρ is $\nabla\rho = 1$ at any point $(\lambda, c) \in X$, ρ does not have any critical point and hence there is no critical value of ρ in $B(\lambda_0, r)$. Then by Inverse Function Theorem, there exists a neighborhood N of (λ_0, c) and a neighborhood M of$\lambda_0 = \rho\big((\lambda_0, c)\big)$such that $\rho : N \to M$ is a bijection and $\rho^{-1}: M \to N$ is analytic. We can choose $r > 0$ small enough so that $M = B(\lambda_0, r)$. Then

$$\rho^{-1} : B(\lambda_0, r) \to X,\ \ \lambda \longmapsto (\lambda, c)$$

is analytic. Thus there exists a map $\rho^{-1} : B(\lambda_0, r) \to N$ given by $\lambda \longmapsto c_\lambda \in \mathrm{Crit}(f_\lambda)$ which is analytic.

Define

$$\varphi :\ B(\lambda_0, r) \to \hat{U} \text{by} \quad \varphi(\lambda) = f_\lambda^{N(\lambda_0)}(c_\lambda) \in \hat{U}.$$

Then φ is continuous and $\varphi(\lambda_0) \in \hat{U}$, which imply that$\varphi^{-1}(\hat{U})$is open and $\lambda_0 \in \varphi^{-1}\big(\hat{U}\big) \subseteq K$. Now it is enough to show that $\varphi^{-1}\big(\hat{U}\big) \subseteq D_0$.

Let $\gamma \in \varphi^{-1}\big(\hat{U}\big)$ be arbitrary. Then $\varphi(\gamma) \in \hat{U}$implies

$$f_\gamma^{N(\lambda_0)}(c_\gamma) \in \hat{U} \Rightarrow\ f_\gamma^n\left(f_\gamma^{N(\lambda_0)}(c_\gamma)\right) \in f_\gamma^n(\hat{U}) \subseteq \hat{U}.$$

Since $\infty \in h\big(B(0, \delta)\big) = \hat{U}$ and h is a Möbius transformation, we have$\hat{U} = B(\infty, \frac{1}{\delta})$. Now$f_\lambda : B\left(\infty, \frac{1}{\delta}\right) \to B\left(\infty, \frac{1}{\delta}\right)$is an analytic function with$f_\lambda(\infty) = \infty$ and $f_\lambda\big(\varphi(\lambda)\big) \in B\left(\infty, \frac{1}{\delta}\right)$for all $\lambda \in \varphi^{-1}\left(B\left(\infty, \frac{1}{\delta}\right)\right)$, then by Schwarz Lemma, we get

$$\lim_{n\to\infty} f_\gamma^n(\widehat{U}) = \lim_{n\to\infty} f_\gamma^n\left(B\left(\infty, \quad \frac{1}{\delta}\right)\right) = \infty,$$

hence $\gamma \in D_0$ and so D_0 is open.

Since f_λ does not have any finite attracting or any non-zero parabolic periodic point for each $\boldsymbol{\lambda}\in D_0$ and the Julia set $J(f_\lambda)$ does not contain any critical point, so we get the following corollary [1].

Corollary 2.2 The only non-zero finite fixed point of f_λ, $-\frac{1}{\lambda}$, is a repelling fixed point with multiplier $\left|1-\frac{1}{\lambda}\right| > 1$.

By Corollary 2.2, since each $\boldsymbol{\lambda}\in D_0$ satisfies $\left|1-\frac{1}{\lambda}\right| > 1$, we see that D_0 is contained in some left half-plane.

Corollary 2.3 The set $D_0 \subsetneq \left\{\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}\setminus\{0\} : \lambda_1 < \frac{1}{2} \text{ and } \lambda_2 \in \mathbb{R}\right\}$.

Proof.Let $\lambda = \lambda_1 + i\lambda_2$. Then we have $\frac{1}{\lambda} = \frac{\lambda_1 - i\lambda_2}{\lambda_1^2+\lambda_2^2}$. Since $\left|1-\frac{1}{\lambda}\right| > 1$ so we get $\left|1-\frac{\lambda_1 - i\lambda_2}{\lambda_1^2+\lambda_2^2}\right| > 1$, which implies

$$\left(1-\frac{\lambda_1}{\lambda_1^2+\lambda_2^2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1^2+\lambda_2^2}\right)^2 > 1.$$

Hence the solution to the inequality above is for any $\lambda = \lambda_1 + i\lambda_2 \in D_0$, $\lambda_1 < \frac{1}{2}$ for any $\lambda_2 \in \mathbb{R}$.

3. The Family of Cubic Polynomials $\{f_{a,b}\}_{a,b\in\mathbb{C}\setminus\{0\}}$

Consider the family of cubic polynomials $\{f_{a,b}\}_{a,b\in\mathbb{C}\setminus\{0\}}$, where

$$f_{a,b}(z) = z(1-az-bz^2).$$

We will study the dynamics of $f_{a,b}$ via topological conjugacy of the function. Since behavior of two topologically conjugate dynamical systems are same, if we know the dynamics of one function, then we can trivially solve the other.

3.1 Topological Conjugacy

Definition 3.1 [1] Two functions f_1 and f_2 are topologically conjugate to one another if there exists a homeomorphism h so that $f_2 = h\circ f_1 \circ h^{-1}$.

Theorem 3.2 If $h(z) = az$, then $g_{a,b}(z) = \left(h \circ f_{a,b}(z) \circ h^{-1}\right)(z) = z - z^2 - \frac{b}{a^2}z^3$, that is, $g_{a,b}$ and $f_{a,b}$ are topologically conjugate via the Möbius transformation $h(z) = az$.

The conjugacy is an equivalent relation which respects iterations[1], that is,

$$\text{if } f_2 = h\circ f_1 \circ h^{-1}, \text{ then } f_2^n = h\circ f_1^n \circ h^{-1} \text{ for all } n \geq 1.$$

If two polynomials are topologically conjugate, then they share equivalent topological and geometrical properties. Since $f_{a,b}$ and $g_{a,b}$ are conjugate, so we can study the dynamics of the family

$$g_{a,b}(z) = z - z^2 - \frac{b}{a^2}z^3$$

for $a, b \in \mathbb{C} \setminus \{0\}$ to understand the family $f_{a,b}$ .Notice that

$$g'_{a,b}(z) = 1 - 2z - \frac{3b}{a^2}z^2 \ ,$$

and the finite solutions to the equation $g_{a,b}(z) = z$ are 0 and $-\frac{a^2}{b}$ with multiplicities 2 and 1, respectively. Also, we have

$$g'_{a,b}(0) = 1 \quad \text{and} \quad g'_{a,b}\left(-\frac{a^2}{b}\right) = 1 - \frac{a^2}{b}.$$

Thus, 0 is a parabolic fixed point of $g_{a,b}$ with multiplicity 2 and $-\frac{a^2}{b}$is the other finite fixed point with multiplier$\left|1 - \frac{a^2}{b}\right|$. So, there is only one petal at the parabolic fixed point 0 for all $a, b \in \mathbb{C} \setminus \{0\}$ (see [4]).

Define the function

$$\kappa : (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C} \setminus \{0\}$$
$$\kappa\big((a,b)\big) = \frac{b}{a^2}.$$

Corollary 3.3 The function $\kappa : (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C} \setminus \{0\}$is holomorphic.

Proof. We know that if $U, V \subseteq \mathbb{C}$ are open, then the set $\Omega = U \times V \subseteq \mathbb{C}^2$is open and a complex valued function $f = f(z_1 , z_2)$ on Ω is defined to be holomorphic if it is holomorphic in each variable separately, that is, if for each $z_1 \in U$ and each$z_2 \in V$, the functions $\omega \longmapsto f(\omega, z_2)$ and $\xi \longmapsto f(z_1, \xi)$ are holomorphic provided the open subset of the complex plane where these functions are defined is non-empty. Note that,$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$is open because the complement of$\mathbb{C}^*$, which is $\{0\}$,is closed. Otherwise, if $\{0\}$ is open, then $\forall\ \varepsilon > 0, \exists\ z \neq 0$, so that$|z - 0| < \varepsilon \wedge z \in \{0\}$, which is impossible. Now, for each $b \in \mathbb{C}^*$, the function $a \longmapsto \kappa\big((a,b)\big)$is a rational function, and for each $a \in \mathbb{C}^*$, the function $b \longmapsto \kappa\big((a,b)\big)$is a polynomial function, and hence are holomorphic on the set $\mathbb{C}^*$. Hence, the function $\kappa\big((a,b)\big)$is holomorphic on$(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.

Since every holomorphic function is continuous, soκ is continuous on$(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. This means, the inverse image of an open set under κ is open. Since $D_0 \subseteq \mathbb{C}^*$is open, then the inverse image of D_0 under κ is open. Note that here we are taking the inverse image of a set, so there is no concern about the function not being one-to-one.

Define $\widetilde{D_0} = \kappa^{-1}(D_0)$. Then the set $\widetilde{D_0}$ is open, and for each$(a,b) \in \widetilde{D_0}$, we have $\boldsymbol{\lambda = \frac{b}{a^2}}$. Thus for each $(a,b) \in \widetilde{D_0}$with$\boldsymbol{\lambda = \frac{b}{a^2}}$, the function f_λ and $g_{a,b}$ have equivalent topological and

geometrical properties. So, $g_{a,b}$ is a parabolic polynomial with a parabolic fixed point 0 and $-\frac{a^2}{b}$ is a repelling fixed point of $g_{a,b}$ with multiplier $\left|1-\frac{a^2}{b}\right| > 1$. For each $(a,b) \in \widetilde{D_0}$, $J(g_{a,b})$ is disconnected. Using conjugacy, we can say that $f_{a.b}$ is parabolic and the Julia set $J(f_{a,b})$ is disconnected. Moreover, for each $\kappa^{-1}(\lambda) = (a,b) \in \widetilde{D_0}$, the Hausdorff dimension function $\widetilde{D_0} \ni (a,b) \longmapsto \mathrm{HD}\left(J(g_{a,b})\right) \in \mathbb{R}$ is real analytic, and hence the function $\widetilde{D_0} \ni (a,b) \longmapsto \mathrm{HD}\left(J(f_{a,b})\right) \in \mathbb{R}$ is real analytic because of conjugacy property. Hence, weconclude the paper by the following theorem.

Theorem 3.4 $\widetilde{D_0} \ni (a,b) \longmapsto \mathrm{HD}\left(J(f_{a,b})\right) \in \mathbb{R}$ is real analytic, where $\widetilde{D_0} = \kappa^{-1}(D_0)$.

REFERENCES

[1] A. F. Beardon, Iteration of Rational Functions, Complex Analytic Dynamical Systems.Springer-Verlag, 1991. ISBN: 978-0-387-95151-5

[2] M. Denker, M. Urbanski, Hausdorand conformal measures on Julia sets with a rationally indi erent periodic point, J. London Math. Soc. 43 (1991), 107-118. 1

[3] H. Akter, M. Urba_nski, Real analyticity of Hausdor_ dimension of Julia sets of parabolic Polynomials $f_\lambda(z) = z(1-z-\lambda z^2)$, Illinois J. Math. 55 (2011), no. 1, 157-184.

[4] J. Milnor, Dynamics in One Complex Variable, Introductory lectures. Friedr. Vieweg & Sohn, Braunschweig, 1999. viii+257 pp. ISBN: 3-528-03130-1

[5] D. Mauldin, M. Urbanski, Parabolic iterated function systems, Ergod. Th. & Dynam. Sys. 20 (2000), 1423-1447.

[6] D. Mauldin, M. Urbanski, Fractal measures for parabolic IFS, Adv. in Math. 168 (2002), 225-253.

[7] Curtis T. McMullen, Hausdorffdimension and conformal dynamics, III: Computation of dimension, American Journal of Mathematics, Johns Hopkins University Press 120 (1998), no. 4, 691-721. 2.2