Question No.1

Problem 1. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that
a) $f(x) \leq f(b)$ for all $a < x < b$;
b) $f(a) = f(b)$.

(José Luis Díaz-Barrero, Barcelona)

Solution. (a) We prove by contradiction. Suppose that exists a point $c \in (a, b)$ such that $f(c) > f(b)$.

By Weierstrass' theorem, f has a maximal value m on $[c, b]$; this value is attained at some point $d \in [c, b]$. Since $f(d) = \max_{[c,b]} f \geq f(c) > f(b)$, we have $d \neq b$, so $d \in [c, b) \subset (a, b)$. The point d, lying in (a, b), is a shadow point, therefore $f(y) > f(d)$ for some $y > d$. From combining our inequalities we get $f(y) > f(d) > f(b)$.

Case 1: $y > b$. Then $f(y) > f(b)$ contradicts the assumption that b is not a shadow point.

Case 2: $y \leq b$. Then $y \in (d, b] \subset [c, b]$, therefore $f(y) > f(d) = m = \max_{[c,b]} f \geq f(y)$, contradiction again.

(b) Since $a < b$ and a is not a shadow point, we have $f(a) \geq f(b)$.

By part (a), we already have $f(x) \leq f(b)$ for all $x \in (a, b)$. By the continuity at a we have

$$f(a) = \lim_{x \to a+0} f(x) \leq \lim_{x \to a+0} f(b) = f(b)$$

Hence we have both $f(a) \geq f(b)$ and $f(a) \leq f(b)$, so $f(a) = f(b)$.

Question No.2

Problem 2. Does there exist a real 3×3 matrix A such that $\mathrm{tr(A)} = 0$ and $A^2 + A^t = I$? ($\mathrm{tr(A)}$ denotes the trace of A, A^t is the transpose of A, and I is the identity matrix.)

(Moubinool Omarjee, Paris)

Solution. The answer is NO.

Suppose that $\mathrm{tr(A)} = 0$ and $A^2 + A^t = I$. Taking the transpose, we have

$$A = I - (A^2)^t = I - (A^t)^2 = I - (I - A^2)^2 = 2A^2 - A^4,$$

$$A^4 - 2A^2 + A = 0.$$

The roots of the polynomial $x^4 - 2x^2 + x = x(x-1)(x^2 + x - 1)$ are $0, 1, \frac{-1 \pm \sqrt{5}}{2}$ so these numbers can be the eigenvalues of A; the eigenvalues of A^2 can be $0, 1, \frac{1 \pm \sqrt{5}}{2}$.

By $\mathrm{tr(A)} = 0$, the sum of the eigenvalues is 0, and by $\mathrm{tr(A^2)} = \mathrm{tr(I - A^t)} = 3$ the sum of squares of the eigenvalues is 3. It is easy to check that this two conditions cannot be satisfied simultaneously.

Question No.3

Problem 3. Let p be a prime number. Call a positive integer n *interesting* if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials f and g with integer coefficients.
a) Prove that the number $p^p - 1$ is interesting.
b) For which p is $p^p - 1$ the minimal interesting number?

(Eugene Goryachko and Fedor Petrov, St. Petersburg)

Solution. (a) Let's reformulate the property of being interesting: n is interesting if $x^n - 1$ is divisible by $x^p - x + 1$ in the ring of polynomials over $\mathbb{F}_p$ (the field of residues modulo p). All further congruences are modulo $x^p - x + 1$ in this ring. We have $x^p \equiv x - 1$, then $x^{p^2} = (x^p)^p \equiv (x-1)^p \equiv x^p - 1 \equiv x - 2$, $x^{p^3} = (x^{p^2})^p \equiv (x-2)^p \equiv x^p - 2^p \equiv x - 2^p - 1 \equiv x - 3$ and so on by Fermat's little theorem, finally $x^{p^p} \equiv x - p \equiv x$,

$$x(x^{p^p - 1} - 1) \equiv 0.$$

Since the polynomials $x^p - x + 1$ and x are coprime, this implies $x^{p^p - 1} - 1 \equiv 0$.

Question No.4

Problem 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence with $\frac{1}{2} < a_n < 1$ for all $n \geq 0$. Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \qquad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \geq 0).$$

What are the possible values of $\lim_{n\to\infty} x_n$? Can such a sequence diverge?

Johnson Olaleru, Lagos

Solution 1. We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have $(1 - x_n) \to 0$ and therefore $x_n \to 1$.

The case $n = 0$ is true since $\frac{1}{2} < x_0 = a_0 < 1$.

Supposing that the induction hypothesis holds for n, from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n).$$

By

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < \frac{1 - \frac{1}{2}}{1 + 0} = \frac{1}{2}$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and $x_n \to 1$.

Question No.5

Problem 3. Determine the value of

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \cdot \ln\left(1 + \frac{1}{2n}\right) \cdot \ln\left(1 + \frac{1}{2n+1}\right).$$

Gerhard Woeginger, Utrecht

Solution. Define $f(n) = \ln(\frac{n+1}{n})$ for $n \geq 1$, and observe that $f(2n) + f(2n+1) = f(n)$. The well-known inequality $\ln(1 + x) \leq x$ implies $f(n) \leq 1/n$. Furthermore introduce

$$g(n) \;=\; \sum_{k=n}^{2n-1} f^3(k) \;<\; n\, f^3(n) \;\leq\; 1/n^2.$$

Then

$$\begin{aligned} g(n) - g(n+1) &= f^3(n) - f^3(2n) - f^3(2n+1) \\ &= (f(2n) + f(2n+1))^3 - f^3(2n) - f^3(2n+1) \\ &= 3\,(f(2n) + f(2n+1))\; f(2n)\, f(2n+1) \\ &= 3\, f(n)\, f(2n)\, f(2n+1), \end{aligned}$$

therefore

$$\sum_{n=1}^{N} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3} \sum_{n=1}^{N} g(n) - g(n+1) \;=\; \frac{1}{3}\,(g(1) - g(N+1)).$$

Since $g(N+1) \to 0$ as $N \to \infty$, the value of the considered sum hence is

$$\sum_{n=1}^{\infty} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3} g(1) \;=\; \frac{1}{3}\ln^3(2).$$

Question No.6

Problem 4. Let $f(x)$ be a polynomial with real coefficients of degree n. Suppose that $\dfrac{f(k)-f(m)}{k-m}$ is an integer for all integers $0 \le k < m \le n$. Prove that $a-b$ divides $f(a)-f(b)$ for all pairs of distinct integers a and b.

Fedor Petrov, St. Petersburg

Solution 1. We need the following

Lemma. Denote the least common multiple of $1, 2, \ldots, k$ by $L(k)$, and define

$$h_k(x) = L(k) \cdot \binom{x}{k} \qquad (k = 1, 2, \ldots).$$

Then the polynomial $h_k(x)$ satisfies the condition, i.e. $a-b$ divides $h_k(a)-h_k(b)$ for all pairs of distinct integers a, b.

Proof. It is known that

$$\binom{a}{k} = \sum_{j=0}^{k} \binom{a-b}{j}\binom{b}{k-j}.$$

(This formula can be proved by comparing the coefficient of x^k in $(1+x)^a$ and $(1+x)^{a-b}(1+x)^b$.) From here we get

$$h_k(a) - h_k(b) = L(K)\left(\binom{a}{k} - \binom{b}{k}\right) = L(K)\sum_{j=1}^{k}\binom{a-b}{j}\binom{b}{k-j} = (a-b)\sum_{j=1}^{k}\frac{L(k)}{j}\binom{a-b-1}{j-1}\binom{b}{k-j}.$$

On the right-hand side all fractions $\frac{L(k)}{j}$ are integers, so the right-hand side is a multiple of (a,b). The lemma is proved.

Expand the polynomial f in the basis $1, \binom{x}{1}, \binom{x}{2}, \ldots$ as

$$f(x) = A_0 + A_1\binom{x}{1} + A_2\binom{x}{2} + \cdots + A_n\binom{x}{n}. \tag{1}$$

We prove by induction on j that A_j is a multiple of $L(j)$ for $1 \le j \le n$. (In particular, A_j is an integer for $j \ge 1$.) Assume that $L(j)$ divides A_j for $1 \le j \le m-1$. Substituting m and some $k \in \{0, 1, \ldots, m-1\}$ in (1),

$$\frac{f(m)-f(k)}{m-k} = \sum_{j=1}^{m-1}\frac{A_j}{L(j)} \cdot \frac{h_j(m)-h_j(k)}{m-k} + \frac{A_m}{m-k}.$$

Question No.7

Problem 1. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

(Proposed by Fedor Duzhin, Nanyang Technological University)

Solution 1. The statement is true for $n = 1$, because $p(0) = p(1) = 1$ and the only partition of 1 contains the term 1. In the rest of the solution we assume $n \geq 2$.

Let $\mathcal{P}_n = \{(a_1, \ldots, a_k) : k \in \mathbb{N},\ a_1 \geq \ldots \geq a_k,\ a_1 + \ldots + a_k = n\}$ be the set of partitions of n, and let $\mathcal{Q}_n = \{(a_1, \ldots, a_k) \in \mathcal{P}_n : a_k = 1\}$ the set of those partitions of n that contain the term 1. The set of those partitions of n that do not contain 1 as a term, is $\mathcal{P}_n \setminus \mathcal{Q}_n$. We have to prove that $|P_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}|$.

Define the map $\varphi\colon \mathcal{P}_{n-1} \to \mathcal{Q}_n$ as

$$\varphi(a_1, \ldots, a_k) = (a_1, \ldots, a_k, 1).$$

This is a partition of n containing 1 as a term (so indeed $\varphi(a_1, \ldots, a_k) \in \mathcal{Q}_n$). Moreover, each partition $(a_1, \ldots, a_k, 1) \in \mathcal{Q}_n$ uniquely determines $(a_1, \ldots, a_k)$. Therefore the map φ is a bijection between the sets $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$. Then $|\mathcal{P}_{n-1}| = |\mathcal{Q}_n|$. Since $\mathcal{Q}_n \subset \mathcal{P}_n$,

$$|\mathcal{P}_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}| = p(n) - p(n-1).$$

Question No.8

Problem 2. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strickly positive real numbers of the main diagonal.

(Proposed by Ilya Bogdanov and Grigoriy Chelnokov, MIPT, Moscow)

Solution. For $n = 1$ the only matrix is (0) with rank 0. For $n = 2$ the determinant of such a matrix is negative, so the rank is 2. We show that for all $n \geq 3$ the minimal rank is 3.

Notice that the first three rows are linearly independent. Suppose that some linear combination of them, with coefficients c_1, c_2, c_3, vanishes. Observe that from the first column one deduces that c_2 and c_3 either have opposite signs or both zero. The same applies to the pairs (c_1, c_2) and (c_1, c_3). Hence they all must be zero.

It remains to give an example of a matrix of rank (at most) 3. For example, the matrix

$$\begin{pmatrix} 0^2 & 1^2 & 2^2 & \ldots & (n-1)^2 \\ (-1)^2 & 0^2 & 1^2 & \ldots & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-n+1)^2 & (-n+2)^2 & (-n+3)^2 & \ldots & 0^2 \end{pmatrix} = \Big((i-j)^2\Big)_{i,j=1}^{n} =$$

$$= \begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \ldots, 1) - 2 \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1, 2, \ldots, n) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \ldots, n^2)$$

is the sum of three matrices of rank 1, so its rank cannot exceed 3.

Question No.9

Problem 1. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}\,x^{2011} + \ldots + a_1\,x + a_0\,.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.

(a) Which of the players has a winning strategy if $m(x) = x - 2012$?

(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

(Proposed by Fedor Duzhin, Nanyang Technological University)

Solution. We show that Homer has a winning strategy in both part (a) and part (b).

(a) Notice that the last move is Homer's, and only the last move matters. Homer wins if and only if $f(2012) = 0$, i.e.

$$2012^{2012} + a_{2011}\,2012^{2011} + \ldots + a_k\,2012^k + \ldots + a_1\,2012 + a_0 \;= 0. \qquad (1)$$

Suppose that all of the coefficients except for a_k have been assigned values. Then Homer's goal is to establish (1) which is a linear equation on a_k. Clearly, it has a solution and hence Homer can win.

(b) Define the polynomials

$$g(y) = a_0 + a_2y + a_4y^2 + \ldots + a_{2010}y^{1005} + y^{1006} \quad \text{and} \quad h(y) = a_1 + a_3y + a_5y^2 + \ldots + a_{2011}y^{1005},$$

so $f(x) = g(x^2) + h(x^2) \cdot x$. Homer wins if he can achieve that $g(y)$ and $h(y)$ are divisible by $y + 1$, i.e. $g(-1) = h(-1) = 0$.

Notice that both $g(y)$ and $h(y)$ have an even number of undetermined coefficients in the beginning of the game. A possible strategy for Homer is to follow Albert: whenever Albert assigns a value to a coefficient in g or h, in the next move Homer chooses the value for a coefficient in the same polynomial. This way Homer defines the last coefficient in g and he also chooses the last coefficient in h. Similarly to part (a), Homer can choose these two last coefficients in such a way that both $g(-1) = 0$ and $h(-1) = 0$ hold.

Question No.10

Problem 2. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{na_n^2}{1+(n+1)a_n} \quad \text{for } n \geq 1.$$

Show that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

(Proposed by Christophe Debry, KU Leuven, Belgium)

Solution. Observe that

$$ka_k = \frac{(1+(k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1} \quad \text{for all } k \geq 1,$$

and hence

$$\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{n}(ka_k - (k+1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1} \qquad (1)$$

for all $n \geq 0$.

By (1) we have $\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} < 1$. Since all terms are positive, this implies that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ is convergent. The sequence of terms, $\frac{a_{k+1}}{a_k}$ must converge to zero. In particular, there is an index n_0 such that $\frac{a_{k+1}}{a_k} < \frac{1}{2}$ for $n \geq n_0$. Then, by induction on n, we have $a_n < \frac{C}{2^n}$ with some positive constant C. From $na_n < \frac{Cn}{2^n} \to 0$ we get $na_n \to 0$, and therefore

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \Big(1 - (n+1)a_{n+1}\Big) = 1.$$

Remark. The inequality $a_n \leq \frac{1}{2^n}$ can be proved by a direct induction as well.

Question No.11

Solution 1. Consider a positive integer n with $n!+1 \big| (2012n)!$. It is well-known that for arbitrary nonnegative integers $a_1, \ldots, a_k$, the number $(a_1 + \ldots + a_k)!$ is divisible by $a_1! \cdot \ldots \cdot a_k!$. (The number of sequences consisting of a_1 digits 1, $\ldots$, a_k digits k, is $\frac{(a_1+\ldots+a_k)!}{a_1!\cdot\ldots\cdot a_k!}$.) In particular, $(n!)^{2012}$ divides $(2012n)!$.

Since $n!+1$ is co-prime with $(n!)^{2012}$, their product $(n!+1)(n!)^{2012}$ also divides $(2012n)!$, and therefore

$$(n!+1)\cdot(n!)^{2012} \le (2012n)!\,.$$

By the known inequalities $\left(\frac{n+1}{e}\right)^n < n! \le n^n$, we get

$$\left(\frac{n}{e}\right)^{2013n} < (n!)^{2013} < (n!+1)\cdot(n!)^{2012} \le (2012n)! < (2012n)^{2012n}$$

$$n < 2012^{2012}e^{2013}.$$

Therefore, there are only finitely many such integers n.

Remark. Instead of the estimate $\left(\frac{n+1}{e}\right)^n < n!$, we may apply the *Multinomial theorem*:

$$(x_1 + \cdots + x_\ell)^N = \sum_{k_1+\ldots+k_\ell=N} \frac{N!}{k_1!\cdot\ldots\cdot k_\ell!} x_1^{k_1}\ldots x_\ell^{k_\ell}.$$

Applying this to $N = 2012n$, $\ell = 2012$ and $x_1 = \ldots = x_\ell = 1$,

$$\frac{(2012n)!}{(n!)^{2012}} < (\underbrace{1+1+\ldots+1}_{2012})^{2012n} = 2012^{2012n},$$

$$n! < n!+1 \le \frac{(2012n)!}{(n!)^{2012}} < 2012^{2012n}.$$

On the right-hand side we have a geometric progression which increases slower than the factorial on the left-hand side, so this is true only for finitely many n.

Question No.12

Problem 4. Let $n \geq 2$ be an integer. Find all real numbers a such that there exist real numbers x_1, $\ldots$, x_n satisfying

$$x_1(1-x_2) = x_2(1-x_3) = \ldots = x_{n-1}(1-x_n) = x_n(1-x_1) = a. \quad (1)$$

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Solution. Throughout the solution we will use the notation $x_{n+1} = x_1$.

We prove that the set of possible values of a is

$$\left(-\infty, \frac{1}{4}\right] \bigcup \left\{ \frac{1}{4\cos^2 \frac{k\pi}{n}};\ k \in \mathbb{N},\ 1 \leq k < \frac{n}{2} \right\}.$$

In the case $a \leq \frac{1}{4}$ we can choose x_1 such that $x_1(1-x_1) = a$ and set $x_1 = x_2 = \ldots = x_n$. Hence we will now suppose that $a > \frac{1}{4}$.

The system (1) gives the recurrence formula

$$x_{i+1} = \varphi(x_i) = 1 - \frac{a}{x_i} = \frac{x_i - a}{x_i}, \quad i = 1, \ldots, n.$$

The fractional linear transform φ can be interpreted as a projective transform of the real projective line $\mathbb{R} \cup \{\infty\}$; the map φ is an element of the group $\mathrm{PGL}_2(\mathbb{R})$, represented by the linear transform $M = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$. (Note that $\det M \neq 0$ since $a \neq 0$.) The transform φ^n can be represented by M^n. A point $[u, v]$ (written in homogenous coordinates) is a fixed point of this transform if and only if $(u, v)^T$ is an eigenvector of M^n. Since the entries of M^n and the coordinates u, v are real, the corresponding eigenvalue is real, too.

The characteristic polynomial of M is $x^2 - x + a$, which has no real root for $a > \frac{1}{4}$. So M has two conjugate complex eigenvalues $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{4a-1}i)$. The eigenvalues of M^n are $\lambda_{1,2}^n$, they are real if and only if $\arg \lambda_{1,2} = \pm\frac{k\pi}{n}$ with some integer k; this is equivalent with

$$\pm\sqrt{4a-1} = \tan\frac{k\pi}{n},$$
$$a = \frac{1}{4}\left(1 + \tan^2 \frac{k\pi}{n}\right) = \frac{1}{4\cos^2 \frac{k\pi}{n}}.$$

If $\arg \lambda_1 = \frac{k\pi}{n}$ then $\lambda_1^n = \lambda_2^n$, so the eigenvalues of M^n are equal. The eigenvalues of M are distinct, so M and M^n have two linearly independent eigenvectors. Hence, M^n is a multiple of the identity. This means that the projective transform φ^n is the identity; starting from an arbitrary point $x_1 \in \mathbb{R} \cup \{\infty\}$, the cycle $x_1, x_2, \ldots, x_n$ closes at $x_{n+1} = x_1$. There are only finitely many cycles $x_1, x_2, \ldots, x_n$ containing the point ∞; all other cycles are solutions for (1).

Remark. If we write $x_j = P + Q\tan t_j$ where P, Q and $t_1, \ldots, t_n$ are real numbers, the recurrence relation re-writes as

$$(P + Q\tan t_j)(1 - P - Q\tan t_{j+1}) = a$$
$$(1-P)Q\tan t_j - PQ\tan t_{j+1} = a + P(P-1) + Q^2 \tan t_j \tan t_{j+1} \quad (j = 1, 2, \ldots, n).$$

In view of the identity $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$, it is reasonable to choose $P = \frac{1}{2}$, and $Q = \sqrt{a - \frac{1}{4}}$. Then the recurrence leads to

$$t_j - t_{j+1} \equiv \arctan\sqrt{4a-1} \pmod{\pi}.$$

Question No.13

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(Proposed by Pavel Kozhevnikov, MIPT, Moscow)

Solution. The transforms given by A and B strictly increase the length of every nonzero vector, this can be seen easily in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product AB also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than -1.

Question No.14

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

(Proposed by Karen Keryan, Yerevan State University, Yerevan, Armenia)

Solution. Let $g(x) = f(x)\cos x$. Since $g(-\pi/2) = g(0) = g(\pi/2) = 0$, by Rolle's theorem there exist some $\xi_1 \in (-\pi/2, 0)$ and $\xi_2 \in (0, \pi/2)$ such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Now consider the function

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x)\cos x - f(x)\sin x}{\cos^2 x}.$$

We have $h(\xi_1) = h(\xi_2) = 0$, so by Rolle's theorem there exist $\xi \in (\xi_1, \xi_2)$ for which

$$\begin{aligned}
0 = h'(\xi) &= \frac{g''(\xi)\cos^2 \xi + 2\cos\xi \sin\xi g'(\xi)}{\cos^4 \xi} = \\
&= \frac{(f''(\xi)\cos\xi - 2f'(\xi)\sin\xi - f(\xi)\cos\xi)\cos\xi + 2\sin\xi(f'(\xi)\cos\xi - f(\xi)\sin\xi)}{\cos^3 \xi} = \\
&= \frac{f''(\xi)\cos^2\xi - f(\xi)(\cos^2\xi + 2\sin^2\xi)}{\cos^3\xi} = \frac{1}{\cos\xi}(f''(\xi) - f(\xi)(1 + 2\tan^2\xi)).
\end{aligned}$$

The last yields the desired equality.

Question No.15

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(Proposed by Oleksandr Rybak, Kiev, Ukraine)

Solution. We prove that for any $n \geq 2$ the answer is 6.

First we show that less than 6 trips is not sufficient. In that case the total quantity of students in all trips would not exceed $5n$. A student meets $n - 1$ other students in each trip, so he or she takes part on at least 3 excursions to meet all of his or her $2n - 1$ schoolmates. Hence the total quantity of students during the trips is not less then $6n$ which is impossible.

Now let's build an example for 6 trips.

If n is even, we may divide $2n$ students into equal groups A, B, C, D. Then we may organize the trips with groups (A, B), (C, D), (A, C), (B, D), (A, D) and (B, C), respectively.

If n is odd and divisible by 3, we may divide all students into equal groups E, F, G, H, I, J. Then the members of trips may be the following: (E, F, G), (E, F, H), (G, H, I), (G, H, J), (E, I, J), (F, I, J).

In the remaining cases let $n = 2x + 3y$ be, where x and y are natural numbers. Let's form the groups A, B, C, D of x students each, and E, F, G, H, I, J of y students each. Then we apply the previous cases and organize the following trips: (A, B, E, F, G), (C, D, E, F, H), (A, C, G, H, I), (B, D, G, H, J), (A, D, E, I, J), (B, C, F, I, J).

Question No.16

Problem 4. Let $n \geq 3$ and let $x_1, \dots, x_n$ be nonnegative real numbers. Define $A = \sum_{i=1}^{n} x_i$, $B = \sum_{i=1}^{n} x_i^2$ and $C = \sum_{i=1}^{n} x_i^3$. Prove that

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$p(X) = \prod_{i=1}^{n}(X - x_i) = X^n - AX^{n-1} + \frac{A^2 - B}{2}X^{n-2} - \frac{A^3 - 3AB + 2C}{6}X^{n-3} + \dots.$$

The $(n-3)$th derivative of p has three nonnegative real roots $0 \leq u \leq v \leq w$. Hence,

$$\frac{6}{n!}p^{(n-3)}(X) = X^3 - \frac{3A}{n}X^2 + \frac{3(A^2 - B)}{n(n-1)}X - \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)} = (X-u)(X-v)(X-w),$$

so

$$u + v + w = \frac{3A}{n}, \quad uv + vw + wu = \frac{3(A^2 - B)}{n(n-1)} \quad \text{and} \quad uvw = \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)}.$$

From these we can see that

$$\frac{n^2(n-1)^2(n-2)}{9}\left((n+1)A^2B + (n-2)B^2 - A^4 - (2n-2)AC\right) = \dots =$$
$$= u^2v^2 + v^2w^2 + w^2u^2 - uvw(u+v+w) = uv(u-w)(v-w) + vw(v-u)(w-u) + wu(w-v)(u-v) \geq$$
$$\geq 0 + uw(v-u)(w-v) + wu(w-v)(u-v) = 0.$$

Question No.17

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{\infty} a_n^p$ converges if and only if p is not a prime?

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. The answer is YES. We prove a more general statement; suppose that $N = C \cup D$ is an arbitrary decomposition of N into two disjoint sets. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n^p$ is convergent for $p \in C$ and divergent for $p \in D$.

Define $C_k = C \cap [1, k]$ and $D_k \cap [1, k]$.

Lemma. For every positive integer k there exists a positive integer N_k and a sequence $X_k = (x_{k,1}, \dots, x_{k,N_k})$ of complex numbers with the following properties:

(a) For $p \in D_k$, we have $\left|\sum_{j=1}^{N_k} x_{k,j}^p\right| \ge 1$.

(b) For $p \in C_k$, we have $\sum_{j=1}^{N_k} x_{k,j}^p = 0$; moreover, $\left|\sum_{j=1}^{m} x_{k,j}^p\right| \le \frac{1}{k}$ holds for $1 \le m \le N_k$.

Proof. First we find some complex numbers $z_1 \dots, z_k$ with

$$\sum_{j=1}^{k} z_j^p = \begin{cases} 0 & p \in C_k \\ 1 & p \in D_k \end{cases} \tag{1}$$

As is well-known, this system of equations is equivalent to another system $\sigma_\nu(z_1, \dots, z_k) = w_\nu$ $(\nu = 1, 2, \dots, k)$ where σ_ν is the νth elementary symmetric polynomial, and the constants w_ν are uniquely determined by the Newton-Waring-Girard formulas. Then the numbers $z_1, \dots, z_k$ are the roots of the polynomial $z^k - w_1 z^{k-1} + - \dots + (-1)^k w_k$ in some order.

Now let

$$M = \left\lceil \max_{1 \le m \le k,\ p \in C_k} \left|\sum_{j=1}^{m} z_j^p\right| \right\rceil$$

and let $N_k = k \cdot (kM)^k$. We define the numbers $x_{k,1} \dots, x_{k,N_k}$ by repeating the sequence $(\frac{z_1}{kM}, \frac{z_2}{kM}, \dots, \frac{z_k}{kM})$ $(kM)^k$ times, i.e. $x_{k,\ell} = \frac{z_j}{kM}$ if $\ell \equiv j \pmod{k}$. Then we have

$$\sum_{j=1}^{N_k} x_{k,j}^p = (kM)^k \sum_{j=1}^{k} \left(\tfrac{z_j}{kM}\right)^p = (kM)^{k-p} \sum_{j=1}^{k} z_j^p;$$

then from (1) the properties (a) and the first part of (b) follows immediately. For the second part of (b), suppose that $p \in C_k$ and $1 \le m \le N_k$; then $m = kr + s$ with some integers r and $1 \le s \le k$ and hence

$$\left|\sum_{j=1}^{m} x_{k,j}^p\right| = \left|\sum_{j=1}^{kr} + \sum_{j=kr+1}^{kr+s}\right| = \left|\sum_{j=1}^{s} \left(\frac{z_j}{kM}\right)^p\right| \le \frac{M}{(kM)^p} \le \frac{1}{k}.$$

The lemma is proved.

Now let $S_k = N_1 \dots, N_k$ (we also define $S_0 = 0$). Define the sequence (a) by simply concatenating the sequences $X_1, X_2, \dots$:

$$(a_1, a_2, \dots) = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2}, \dots, x_{k,1}, \dots, x_{k,N_k}, \dots); \tag{1}$$

$$a_{S_k+j} = x_{k+1,j} \quad (1 \le j \le N_{k+1}). \tag{2}$$

If $p \in D$ and $k \ge p$ then

$$\left|\sum_{j=S_k+1}^{S_{k+1}} a_j^p\right| = \left|\sum_{j=1}^{N_{k+1}} x_{k+1,j}^p\right| \ge 1;$$

By Cauchy's convergence criterion it follows that $\sum a_n^p$ is divergent.

If $p \in C$ and $S_u < n \le S_{u+1}$ with some $u \ge p$ then

$$\left|\sum_{j=S_p+1}^{n} a_n^p\right| = \left|\sum_{k=p+1}^{u-1} \sum_{j=1}^{N_k} x_{k,j}^p + \sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| = \left|\sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| \le \frac{1}{u}.$$

Then it follows that $\sum_{n=S_p+1}^{\infty} a_n^p = 0$, and thus $\sum_{n=1}^{\infty} a_n^p = 0$ is convergent.

Question No.18

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\text{trace}(M) = a$ and $\det(M) = b$.

(Proposed by Stephan Wagner, Stellenbosch University)

Solution 1. Let the matrix be

$$M = \begin{bmatrix} x & z \\ z & y \end{bmatrix}.$$

The two conditions give us $x + y = a$ and $xy - z^2 = b$. Since this is symmetric in x and y, the matrix can only be unique if $x = y$. Hence $2x = a$ and $x^2 - z^2 = b$. Moreover, if (x, y, z) solves the system of equations, so does $(x, y, -z)$. So M can only be unique if $z = 0$. This means that $2x = a$ and $x^2 = b$, so $a^2 = 4b$.

If this is the case, then M is indeed unique: if $x + y = a$ and $xy - z^2 = b$, then

$$(x - y)^2 + 4z^2 = (x + y)^2 + 4z^2 - 4xy = a^2 - 4b = 0,$$

so we must have $x = y$ and $z = 0$, meaning that

$$M = \begin{bmatrix} a/2 & 0 \\ 0 & a/2 \end{bmatrix}$$

is the only solution.

Question No.19

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs (α, β) of positive real numbers such that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n^\alpha} = \beta$.

(Proposed by Tomas Barta, Charles University, Prague)

Solution. Let $N_n = \binom{n+1}{2}$ (then a_{N_n} is the first appearance of number n in the sequence) and consider limit of the subsequence

$$b_{N_n} := \frac{\sum_{k=1}^{N_n} a_k}{N_n^\alpha} = \frac{\sum_{k=1}^{n} 1 + \cdots + k}{\binom{n+1}{2}^\alpha} = \frac{\sum_{k=1}^{n} \binom{k+1}{2}}{\binom{n+1}{2}^\alpha} = \frac{\binom{n+2}{3}}{\binom{n+1}{2}^\alpha} = \frac{\frac{1}{6}n^3(1+2/n)(1+1/n)}{(1/2)^\alpha n^{2\alpha}(1+1/n)^\alpha}.$$

We can see that $\lim_{n\to\infty} b_{N_n}$ is positive and finite if and only if $\alpha = 3/2$. In this case the limit is equal to $\beta = \frac{\sqrt{2}}{3}$. So, this pair $(\alpha, \beta) = (\frac{3}{2}, \frac{\sqrt{2}}{3})$ is the only candidate for solution. We will show convergence of the original sequence for these values of α and β.

Let N be a positive integer in $[N_n+1, N_{n+1}]$, i.e., $N = N_n + m$ for some $1 \le m \le n+1$. Then we have

$$b_N = \frac{\binom{n+2}{3} + \binom{m+1}{2}}{\left(\binom{n+1}{2} + m\right)^{3/2}}$$

which can be estimated by

$$\frac{\binom{n+2}{3}}{\left(\binom{n+1}{2} + n\right)^{3/2}} \le b_N \le \frac{\binom{n+2}{3} + \binom{n+1}{2}}{\binom{n+1}{2}^{3/2}}.$$

Since both bounds converge to $\frac{\sqrt{2}}{3}$, the sequence b_N has the same limit and we are done.

Question No.20

Problem 2. Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1\le i<j\le n} a_{ii}a_{jj} \ge \sum_{1\le i<j\le n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

(Proposed by Martin Niepel, Comenius University, Bratislava)

Solution. Eigenvalues of a real symmetric matrix are real, hence the inequality makes sense. Similarly, for Hermitian matrices diagonal entries as well as eigenvalues have to be real.

Since the trace of a matrix is the sum of its eigenvalues, for A we have

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

and consequently

$$\sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ii}a_{jj} = \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j.$$

Therefore our inequality is equivalent to

$$\sum_{i=1}^{n} a_{ii}^2 \le \sum_{i=1}^{n} \lambda_i^2.$$

Matrix A^2, which is equal to A^TA (or A^*A in Hermitian case), has eigenvalues $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$. On the other hand, the trace of A^TA gives the square of the Frobenius norm of A, so we have

$$\sum_{i=1}^{n} a_{ii}^2 \le \sum_{i,j=1}^{n} |a_{ij}|^2 = \mathrm{tr}(A^TA) = \mathrm{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2.$$

The inequality follows, and it is clear that the equality holds for diagonal matrices only.

Remark. Same statement is true for Hermitian matrices.

Question No.21

Problem 3. Let $f(x) = \dfrac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $\left|f^{(n)}(x)\right| < \dfrac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution 1. Putting $f(0) = 1$ we can assume that the function is analytic in $\mathbb{R}$. Let $g(x) = x^{n+1}(f^n(x) - \frac{1}{n+1})$. Then $g(0) = 0$ and

$$g'(x) = (n+1)x^n\left(f^{(n)}(x) - \frac{1}{n+1}\right) + x^{n+1}f^{(n+1)}(x) =$$

$$= x^n\left((n+1)f^{(n)}(x) + xf^{(n+1)}(x) - 1\right) = x^n\left((xf(x))^{(n+1} - 1\right) = x^n(\sin^{(n+1)}(x) - 1) \le 0.$$

Hence $g(x) \le 0$ for $x > 0$. Taking into account that $g'(x) < 0$ for $0 < x < \frac{\pi}{2}$ we obtain the desired (strict) inequality for $x > 0$.

Question No.22

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wrocław University of Technology)

Solution. Multiplying the equation by $(A+B)$ we get

$$I = (A+B)(A+B)^{-1} = (A+B)(A^{-1}+B^{-1}) =$$
$$= AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I + AB^{-1} + BA^{-1} + I$$
$$AB^{-1} + BA^{-1} + I = 0.$$

Let $X = AB^{-1}$; then $A = XB$ and $BA^{-1} = X^{-1}$, so we have $X + X^{-1} + I = 0$; multiplying by $(X - I)X$,

$$0 = (X-I)X \cdot (X + X^{-1} + I) = (X-I) \cdot (X^2 + X + I) = X^3 - I.$$

Hence,

$$X^3 = I$$
$$(\det X)^3 = \det(X^3) = \det I = 1$$
$$\det X = 1$$
$$\det A = \det(XB) = \det X \cdot \det B = \det B.$$

In case of complex matrices the statement is false. Let $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. Obviously $\omega \notin \mathbb{R}$ and $\omega^3 = 1$, so $0 = 1 + \omega + \omega^2 = 1 + \omega + \overline{\omega}$.

Let $A = I$ and let B be a diagonal matrix with all entries along the diagonal equal to either ω or $\overline{\omega} = \omega^2$ such a way that $\det(B) \neq 1$ (if n is not divisible by 3 then one may set $B = \omega I$). Then $A^{-1} = I$, $B^{-1} = \overline{B}$. Obviously $I + B + \overline{B} = 0$ and

$$(A+B)^{-1} = (-\overline{B})^{-1} = -B = I + \overline{B} = A^{-1} + B^{-1}.$$

By the choice of A and B, $\det A = 1 \neq \det B$.

Question No.23

Problem 3. Let $F(0)=0$, $F(1)=\frac{3}{2}$, and $F(n)=\frac{5}{2}F(n-1)-F(n-2)$ for $n\ge 2$.

Determine whether or not $\sum_{n=0}^{\infty}\frac{1}{F(2^n)}$ is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

Solution 1. The characteristic equation of our linear recurrence is $x^2-\frac{5}{2}x+1=0$, with roots $x_1=2$ and $x_2=\frac{1}{2}$. So $F(n)=a\cdot 2^n+b\cdot(\frac{1}{2})^n$ with some constants a,b. By $F(0)=0$ and $F(1)=\frac{3}{2}$, these constants satisfy $a+b=0$ and $2a+\frac{b}{2}=\frac{3}{2}$. So $a=1$ and $b=-1$, and therefore

$$F(n)=2^n-2^{-n}.$$

Observe that

$$\frac{1}{F(2^n)}=\frac{2^{2^n}}{(2^{2^n})^2-1}=\frac{1}{2^{2^n}-1}-\frac{1}{(2^{2^n})^2-1}=\frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1},$$

so

$$\sum_{n=0}^{\infty}\frac{1}{F(2^n)}=\sum_{n=0}^{\infty}\left(\frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1}\right)=\frac{1}{2^{2^0}-1}=1.$$

Hence the sum takes the value 1, which is rational.

Solution 2. As in the first solution we find that $F(n)=2^n-2^{-n}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{F(2^n)} &= \sum_{n=0}^{\infty}\frac{1}{2^{2^n}-2^{-2^n}} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})^{2^n}}{1-(\frac{1}{2})^{2^{n+1}}}\\ &= \sum_{n=0}^{\infty}(\tfrac{1}{2})^{2^n}\sum_{k=0}^{\infty}\left((\tfrac{1}{2})^{2^{n+1}}\right)^k = \sum_{n=0}^{\infty}(\tfrac{1}{2})^{2^n}\sum_{k=0}^{\infty}(\tfrac{1}{2})^{2k\cdot 2^n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(\tfrac{1}{2})^{2^n(2k+1)} = \sum_{m=1}^{\infty}(\tfrac{1}{2})^m = 1.\end{aligned}$$

(Here we used the fact that every positive integer m has a unique representation $m=2^n(2k+1)$ with non-negative integers n and k.)

This shows that the series converges to 1.

Question No.24

Problem 6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < 2.$$

(Proposed by Ivan Krijan, University of Zagreb)

Solution. We prove that

$$\frac{1}{\sqrt{n}\,(n+1)} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}. \tag{1}$$

Multiplying by $\sqrt{n}(n+1)$, the inequality (1) is equivalent with

$$1 < 2(n+1) - 2\sqrt{n(n+1)}$$

$$2\sqrt{n(n+1)} < n + (n+1)$$

which is true by the AM-GM inequality.

Applying (1) to the terms in the left-hand side,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} \right) = 2.$$

Question No.25

Problem 7. Compute

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x\,.$$

(Proposed by Jan Šustek, University of Ostrava)

Solution 1. We prove that

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = 1\,.$$

For $A > 1$ the integrand is greater than 1, so

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x > \frac{1}{A} \int_1^A 1 \mathrm{d}x = \frac{1}{A}(A-1) = 1 - \frac{1}{A}.$$

In order to find a tight upper bound, fix two real numbers, $\delta > 0$ and $K > 0$, and split the interval into three parts at the points $1+\delta$ and $K \log A$. Notice that for sufficiently large A (i.e., for $A > A_0(\delta, K)$ with some $A_0(\delta, K) > 1$) we have $1 + \delta < K \log A < A$.) For $A > 1$ the integrand is decreasing, so we can estimate it by its value at the starting points of the intervals:

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = \frac{1}{A} \left(\int_1^{1+\delta} + \int_{1+\delta}^{K\log A} + \int_{K\log A}^{A} \right) <$$

$$= \frac{1}{A} \left(\delta \cdot A + (K \log A - 1 - \delta) A^{\frac{1}{1+\delta}} + (A - K\log A) A^{\frac{1}{K\log A}} \right) <$$

$$< \frac{1}{A} \left(\delta A + K A^{\frac{1}{1+\delta}} \log A + A \cdot A^{\frac{1}{K\log A}} \right) = \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Hence, for $A > A_0(\delta, K)$ we have

$$1 - \frac{1}{A} < \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x < \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Question No.26

Problem 1. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x \in (a,b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.

(b) Give an example of such a function on $[0,1]$.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. (a) Choose a convergent sequence z_n of zeros and let $c = \lim z_n \in [a,b]$. By the continuity of f we obtain $f(c) = 0$. We want to show that either $c = a$ or $c = b$, so $f(a) = 0$ or $f(b) = 0$; then the statement follows.

If c was an interior point then we would have $f(c) = 0$ and $f'(c) = \lim \dfrac{f(z_n) - f(c)}{z_n - c} = \lim \dfrac{0-0}{z_n - c} = 0$ simultaneously, contradicting the conditions. Hence, $c = a$ or $c = b$.

(b) Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0. \end{cases}$$

This function has zeros at the points $\dfrac{1}{k\pi}$ for $k = 1, 2, \ldots$, and it is continuous at 0 as well.

In $(0,1)$ we have

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

Since $\sin \frac{1}{x}$ and $\cos \frac{1}{x}$ cannot vanish at the same point, we have either $f(x) \neq 0$ or $f'(x) \neq 0$ everywhere in $(0,1)$.

Question No.27

Problem 1. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \le 2.$$

(Proposed by Gerhard J. Woeginger, The Netherlands)

Solution. By interchanging the sums,

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{1 \le n \le k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right).$$

Then we use the upper bound

$$\sum_{k=n}^{\infty} \frac{1}{k^2} \le \sum_{k=n}^{\infty} \frac{1}{k^2 - \frac{1}{4}} = \sum_{k=n}^{\infty} \left(\frac{1}{k - \frac{1}{2}} - \frac{1}{k + \frac{1}{2}} \right) = \frac{1}{n - \frac{1}{2}}$$

and get

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right) < \sum_{n=1}^{\infty} \left(x_n \cdot \frac{1}{n - \frac{1}{2}} \right) = 2 \sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 2.$$

Question No.28

Problem 2. Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x)+f(y) \geq |x-y|$ for all pairs $x,y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. The minimum of $\int_0^1 f$ is $\frac{1}{4}$.

Applying the condition with $0 \leq x \leq \frac{1}{2}$, $y = x + \frac{1}{2}$ we get

$$f(x) + f(x+\tfrac{1}{2}) \geq \tfrac{1}{2}.$$

By integrating,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \big(f(x) + f(x+\tfrac{1}{2})\big)\,\mathrm{d}x \geq \int_0^{1/2} \tfrac{1}{2}\,\mathrm{d}x = \tfrac{1}{4}.$$

On the other hand, the function $f(x) = \left|x - \frac{1}{2}\right|$ satisfies the conditions because

$$|x-y| = \left|\big(x-\tfrac{1}{2}\big) + \big(\tfrac{1}{2}-y\big)\right| \leq \left|x-\tfrac{1}{2}\right| + \left|\tfrac{1}{2}-y\right| = f(x)+f(y),$$

and establishes

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \big(\tfrac{1}{2}-x\big)\,\mathrm{d}x + \int_{1/2}^1 \big(x-\tfrac{1}{2}\big)\,\mathrm{d}x = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}.$$

Question No.29

Problem 5. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\|\leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(Proposed by Ian Morris and Fedor Petrov, St. Petersburg State University)

Solution 1. Let $r = \|A\|$. We have to prove $\|A^n\| \leq \frac{n}{\ln 2} r^{n-1}$.

As is well-known, the matrix norm satisfies $\|XY\| \leq \|X\| \cdot \|Y\|$ for any matrices X, Y, and as a simple consequence, $\|A^k\| \leq \|A\|^k = r^k$ for every positive integer k.

Let $\chi(t) = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_n) = t^n + c_1 t^{n-1} + \dots + c_n$ be the characteristic polynomial of A. From Vieta's formulas we get

$$|c_k| = \left|\sum_{1\leq i_1<\ldots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}\right| \leq \sum_{1\leq i_1<\ldots<i_k\leq n} \left|\lambda_{i_1}\cdots\lambda_{i_k}\right| \leq \binom{n}{k} \quad (k=1,2,\ldots,n)$$

By the Cayley–Hamilton theorem we have $\chi(A)=0$, so

$$\|A^n\| = \|c_1A^{n-1}+\cdots+c_n\| \le \sum_{k=1}^{n}\binom{n}{k}\|A^k\| \le \sum_{k=1}^{n}\binom{n}{k}r^k = (1+r)^n - r^n.$$

Combining this with the trivial estimate $\|A^n\| \le r^n$, we have

$$\|A^n\| \le \min\big(r^n, (1+r)^n - r^n\big).$$

Let $r_0 = \frac{1}{\sqrt[n]{2}-1}$; it is easy to check that the two bounds are equal if $r=r_0$, moreover

$$r_0 = \frac{1}{e^{\ln 2/n}-1} < \frac{n}{\ln 2}.$$

For $r \le r_0$ apply the trivial bound:

$$\|A^n\| \le r^n \le r_0 \cdot r^{n-1} < \frac{n}{\ln 2}r^{n-1}.$$

For $r > r_0$ we have

$$\|A^n\| \le (1+r)^n - r^n = r^{n-1}\cdot\frac{(1+r)^n - r^n}{r^{n-1}}.$$

Notice that the function $f(r) = \frac{(1+r)^n - r^n}{r^{n-1}}$ is decreasing because the numerator has degree $n-1$ and all coefficients are positive, so

$$\frac{(1+r)^n - r^n}{r^{n-1}} < \frac{(1+r_0)^n - r_0^n}{r_0^{n-1}} = r_0\big((1+1/r_0)^n - 1\big) = r_0 < \frac{n}{\ln 2},$$

so $\|A^n\| < \frac{n}{\ln 2} r^{n-1}$.

Solution 2. We will use the following facts which are easy to prove:

- For any square matrix A there exists a unitary matrix U such that UAU^{-1} is upper-triangular.
- For any matrices A, B we have $\|A\| \le \|(A|B)\|$ and $\|B\| \le \|(A|B)\|$ where $(A|B)$ is the matrix whose columns are the columns of A and the columns of B.
- For any matrices A, B we have $\|A\| \le \|\binom{A}{B}\|$ and $\|B\| \le \|\binom{A}{B}\|$ where $\binom{A}{B}$ is the matrix whose rows are the rows of A and the rows of B.
- Adding a zero row or a zero column to a matrix does not change its norm.

We will prove a stronger inequality

$$\|A^n\| \le n\|A\|^{n-1}$$

for any $n \times n$ matrix A whose eigenvalues have absolute value at most 1. We proceed by induction on n. The case $n = 1$ is trivial. Without loss of generality we can assume that the matrix A is upper-triangular. So we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Note that the eigenvalues of A are precisely the diagonal entries. We split A as the sum of 3 matrices, $A = X + Y + Z$ as follows:

$$X = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Denote by A' the matrix obtained from A by removing the first row and the first column:

$$A' = \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots \\ 0 & \cdots & a_{nn} \end{pmatrix}.$$

We have $\|X\| \leq 1$ because $|a_{11}| \leq 1$. We also have

$$\|A'\| = \|Z\| \leq \|Y + Z\| \leq \|A\|.$$

Now we decompose A^n as follows:

$$A^n = XA^{n-1} + (Y + Z)A^{n-1}.$$

We substitute $A = X + Y + Z$ in the second term and expand the parentheses. Because of the following identities:

$$Y^2 = 0, \quad YX = 0, \quad ZY = 0, \quad ZX = 0$$

only the terms YZ^{n-1} and Z^n survive. So we have

$$A^n = XA^{n-1} + (Y + Z)Z^{n-1}.$$

By the induction hypothesis we have $\|A'^{n-1}\| \leq (n-1)\|A'\|^{n-2}$, hence $\|Z^{n-1}\| \leq (n-1)\|Z\|^{n-2} \leq (n-1)\|A\|^{n-2}$. Therefore

$$\|A^n\| \leq \|XA^{n-1}\| + \|(Y + Z)Z^{n-1}\| \leq \|A\|^{n-1} + (n-1)\|Y + Z\|\|A\|^{n-2} \leq n\|A\|^{n-1}.$$

Question No.30

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. By taking squares,

$$A^4 = (A^2)^2 = (A^T)^2 = (A^2)^T = (A^T)^T = A,$$

so

$$A^4 - A = 0;$$

it follows that all eigenvalues of A are roots of the polynomial $X^4 - X$.

The roots of $X^4 - X = X(X^3 - 1)$ are 0, 1 and $\frac{-1\pm\sqrt{3}i}{2}$. In order to verify that these values are possible, consider the matrices

$$A_0 = (0), \quad A_1 = (1), \quad A_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The numbers 0 and 1 are the eigenvalues of the 1×1 matrices A_0 and A_1, respectively. The numbers $\frac{-1\pm\sqrt{3}i}{2}$ are the eigenvalues of A_2; it is easy to check that

$$A_2^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = A_2^T.$$

The matrix A_4 establishes all the four possible eigenvalues in a single matrix.

Remarks. The matrix A_2 represents a rotation by $2\pi/3$.

Question No.31

Remark. The matrix A_2 represents a rotation by $2\pi/3$.

Problem 2. Let $f: \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$\left|f'(x) - f'(y)\right| \le L\left|x - y\right|$$

for all x, y. Prove that

$$\left(f'(x)\right)^2 < 2Lf(x)$$

holds for all x.

(Proposed by Jan Šustek, University of Ostrava)

Solution. Notice that f' satisfies the Lipschitz-property, so f' is continuous and therefore locally integrable.

Consider an arbitrary $x \in \mathbb{R}$ and let $d = f'(x)$. We need to prove $f(x) > \frac{d^2}{2L}$.

If $d = 0$ then the statement is trivial.

If $d > 0$ then the condition provides $f'(x-t) \geq d - Lt$; this estimate is positive for $0 \leq t < \frac{d}{L}$. By integrating over that interval,

$$f(x) > f(x) - f(x - \tfrac{d}{L}) = \int_0^{\frac{d}{L}} f'\big(x-t\big)\mathrm{d}t \geq \int_0^{\frac{d}{L}} (d - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

If $d < 0$ then apply $f'(x+t) \leq d + Lt = -|d| + Lt$ and repeat the same argument as

$$f(x) > f(x) - f(x + \tfrac{|d|}{L}) = \int_0^{\frac{|d|}{L}} \big(-f'(x+t)\big)\mathrm{d}t \geq \int_0^{\frac{|d|}{L}} (|d| - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

Question No.32

Problem 5. Let k and n be positive integers with $n \geq k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0 c_{n-2} = c_1 c_{n-3} = \ldots = c_{n-2} c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

(Proposed by Vsevolod Lev and Fedor Petrov, St. Petersburg State University)

Solution. Let $M = \{z : z^n = 1\}$, $A = \{z \in M : f(z) \neq 0\}$ and $A^{-1} = \{z^{-1} : z \in A\}$. We have to prove $|A| \geq k$.

Claim.

$$A \cdot A^{-1} = M.$$

That is, for any $\eta \in M$, there exist some elements $a, b \in A$ such that $ab^{-1} = \eta$.

Proof. As is well-known, for every integer m,

$$\sum_{z \in M} z^m = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

Question No.33

Problem 6. Let $f:[0;+\infty)\to\mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x)=L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty}\int_0^1 f(nx)\,\mathrm{d}x = L.$$

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution 1. *Case 1: L is finite.* Take an arbitrary $\varepsilon>0$. We construct a number $K\geq 0$ such that $\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| < \varepsilon$.

Since $\lim_{x\to+\infty} f(x)=L$, there exists a $K_1\geq 0$ such that $\left|f(x)-L\right|<\frac{\varepsilon}{2}$ for every $x\geq K_1$. Hence, for $n\geq K_1$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| = \left|\frac{1}{n}\int_0^n f(x)\,\mathrm{d}x - L\right| = \frac{1}{n}\left|\int_0^n (f-L)\right| \leq$$
$$\leq \frac{1}{n}\int_0^n |f-L| = \frac{1}{n}\left(\int_0^{K_1}|f-L| + \int_{K_1}^n |f-L|\right) < \frac{1}{n}\left(\int_0^{K_1}|f-L| + \int_{K_1}^n \frac{\varepsilon}{2}\right) =$$
$$= \frac{1}{n}\int_0^{K_1}|f-L| + \frac{n-K_1}{n}\cdot\frac{\varepsilon}{2} < \frac{1}{n}\int_0^{K_1}|f-L| + \frac{\varepsilon}{2}.$$

If $n\geq K_2=\frac{2}{\varepsilon}\int_0^{K_1}|f-L|$ then the first term is at most $\frac{\varepsilon}{2}$. Then for $x\geq K:=\max(K_1,K_2)$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Case 2: $L=+\infty$. Take an arbitrary real M; we need a $K\geq 0$ such that $\int_0^1 f(nx)\,\mathrm{d}x > M$ for every $x\geq K$.

Since $\lim_{x\to+\infty} f(x)=\infty$, there exists a $K_1\geq 0$ such that $f(x)>M+1$ for every $x\geq K_1$. Hence, for $n\geq 2K_1$ we have

$$\int_0^1 f(nx)\,\mathrm{d}x = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x = \frac{1}{n}\int_0^n f = \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n f\right) =$$
$$= \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n (M+1)\right) = \frac{1}{n}\left(\int_0^{K_1} f - K_1(M+1)\right) + M+1.$$

If $n\geq K_2:=\left|\int_0^{K_1} f - K_1(M+1)\right|$ then the first term is at least -1. For $x\geq K:=\max(K_1,K_2)$ we have $\int_0^1 f(nx)\,\mathrm{d}x > M$.

Case 3: $L=-\infty$. We can repeat the steps in Case 2 for the function $-f$.

Solution 2. Let $F(x) = \int_0^x f$. For $t > 0$ we have

$$\int_0^1 f(tx)\,\mathrm{d}x = \frac{F(t)}{t}.$$

Since $\lim\limits_{t\to\infty} t = \infty$ in the denominator and $\lim\limits_{t\to\infty} F'(t) = \lim\limits_{t\to\infty} f(t) = L$, L'Hospital's rule proves $\lim\limits_{t\to\infty} \frac{F(t)}{t} = \lim\limits_{t\to\infty} \frac{F'(t)}{1} = \lim\limits_{t\to\infty} \frac{f(t)}{1} = L$. Then it follows that $\lim \frac{F(n)}{n} = L$.

Question No.34

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

(Proposed by Snježana Majstorović, University of J. J. Strossmayer in Osijek, Croatia)

Solution. For each $n \in \mathbb{N}$, matrix A_n is symmetric $2^n \times 2^n$ matrix with elements from the set $\{0, 1\}$, so that all elements on the main diagonal are equal to zero. We can write

$$A_n = I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2, \tag{1}$$

where $\otimes$ is binary operation over the space of matrices, defined for arbitrary $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{m\times s}$ as

$$B \otimes C := \begin{bmatrix} b_{11}C & b_{12}C & \ldots & b_{1p}C \\ b_{21}C & b_{22}C & \ldots & b_{2p}C \\ \vdots & & & \\ b_{n1}C & b_{12}C & \ldots & b_{np}C \end{bmatrix}_{nm\times ps}.$$

Lemma 1. If $B \in \mathbb{R}^{n\times n}$ has eigenvalues λ_i, $i = 1, \ldots, n$ and $C \in \mathbb{R}^{m\times m}$ has eigenvalues μ_j, $j = 1, \ldots, m$, then $B \otimes C$ has eigenvalues $\lambda_i\mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If B and C are diagonalizable, then $A \otimes B$ has eigenvectors $y_i \otimes z_j$, with (λ_i, y_i) and (μ_j, z_j) being eigenpairs of B and C, respectively.

Proof 1. Let (λ, y) be an eigenpair of B and (μ, z) an eigenpar of C. Then

$$(B \otimes C)(y \otimes z) = By \otimes Cz = \lambda y \otimes \mu z = \lambda\mu(y \otimes z).$$

If we take (λ, y) to be an eigenpair of A_1 and (μ, z) to be an eigenpair of A_{n-1}, then from (1) and Lemma 1 we get

$$\begin{aligned} A_n(z \otimes y) &= (I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2)(z \otimes y) \\ &= (I_{2^{n-1}} \otimes A_1)(z \otimes y) + (A_{n-1} \otimes I_2)(z \otimes y) \\ &= (\lambda + \mu)(z \otimes y). \end{aligned}$$

So the entire spectrum of A_n can be obtained from eigenvalues of A_{n-1} and A_1: just sum up each eigenvalue of A_{n-1} with each eigenvalue of A_1. Since the spectrum of A_1 is $\sigma(A_1) = \{-1, 1\}$, we get

$$\begin{aligned} \sigma(A_2) &= \{-1 + (-1), -1 + 1, 1 + (-1), 1 + 1\} = \{-2, 0^{(2)}, 2\} \\ \sigma(A_3) &= \{-1 + (-2), -1 + 0, -1 + 0, -1 + 2, 1 + (-2), 1 + 0, 1 + 0, 1 + 2\} = \{-3, (-1)^{(3)}, 1^{(3)}, 3\} \\ \sigma(A_4) &= \{-1 + (-3), -1 + (-1^{(3)}), -1 + 1^{(3)}, -1 + 3, 1 + (-3), 1 + (-1^{(3)}), 1 + 1^{(3)}, 1 + 3\} \\ &= \{-4, (-2)^{(4)}, 0^{(3)}, 2^{(4)}, 4\}. \end{aligned}$$

Inductively, A_n has $n+1$ distinct integer eigenvalues $-n, -n+2, -n+4, \ldots, n-4, n-2, n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, respectively.

Question No.35

Problem 9. Define the sequence $f_1, f_2, \ldots : [0, 1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0,1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1)$ and determine the limit function.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. First of all, the sequence f_n is well defined and it holds that

$$f_{n+1}(x) = e^{\int_0^x f_n(t)\mathrm{d}t}. \tag{2}$$

The mapping $\Phi : C([0, 1)) \to C([0, 1))$ given by

$$\Phi(g)(x) = e^{\int_0^x g(t)\mathrm{d}t}$$

is monotone, i.e. if $f < g$ on $(0, 1)$ then

$$\Phi(f)(x) = e^{\int_0^x f(t)\mathrm{d}t} < e^{\int_0^x g(t)\mathrm{d}t} = \Phi(g)(x)$$

on $(0, 1)$. Since $f_2(x) = e^{\int_0^x 1 mathrmdt} = e^x > 1 = f_1(x)$ on $(0, 1)$, we have by induction $f_{n+1}(x) > f_n(x)$ for all $x \in (0, 1)$, $n \in \mathbb{N}$. Moreover, function $f(x) = \frac{1}{1-x}$ is the unique solution to $f' = f^2$, $f(0) = 1$, i.e. it is the unique fixed point of Φ in $\{\varphi \in C([0, 1)) : \varphi(0) = 1\}$. Since $f_1 < f$ on $(0, 1)$, by induction we have $f_{n+1} = \Phi(f_n) < \Phi(f) = f$ for all $n \in \mathbb{N}$. Hence, for every $x \in (0, 1)$ the sequence $f_n(x)$ is increasing and bounded, so a finite limit exists.

Let us denote the limit $g(x)$. We show that $g(x) = f(x) = \frac{1}{1-x}$. Obviously, $g(0) = \lim f_n(0) = 1$. By $f_1 \equiv 1$ and (2), we have $f_n > 0$ on $[0, 1)$ for each $n \in \mathbb{N}$, and therefore (by (2) again) the function f_{n+1} is increasing. Since f_n, f_{n+1} are positive and increasing also f'_{n+1} is increasing (due to $f'_{n+1} = f_n f_{n+1}$), hence f_{n+1} is convex. A pointwise limit of a sequence of convex functions is convex, since we pass to a limit $n \to \infty$ in

$$f_n(\lambda x + (1 - \lambda)y) \le \lambda f_n(x) + (1 - \lambda) f_n(y)$$

and obtain

$$g(\lambda x+(1-\lambda)y)\leq \lambda g(x)+(1-\lambda)g(y)$$

for any fixed $x,\ y\in[0,1)$ and $\lambda\in(0,1)$. Hence, g is convex, and therefore continuous on $(0,1)$. Moreover, g is continuous in 0, since $1\equiv f_1\leq g\leq f$ and $\lim_{x\to 0+}f(x)=1$. By Dini's Theorem, convergence $f_n\to g$ is uniform on $[0,1-\varepsilon]$ for each $\varepsilon\in(0,1)$ (a monotone sequence converging to a continuous function on a compact interval). We show that Φ is continuous and therefore f_n have to converge to a fixed point of Φ.

In fact, let us work on the space $C([0,1-\varepsilon])$ with any fixed $\varepsilon\in(0,1)$, $\|\cdot\|$ being the supremum norm on $[0,1-\varepsilon]$. Then for a fixed function h and $\|\varphi-h\|<\delta$ we have

$$\sup_{x\in[0,1-\varepsilon]}|\Phi(h)(x)-\Phi(\varphi)(x)|=\sup_{x\in[0,1-\varepsilon]}e^{\int_0^x h(t)\mathrm{d}t}\left|1-e^{\int_0^x \varphi(t)-h(t)\mathrm{d}t}\right|\leq C(e^{\delta}-1)<2C\delta$$

for $\delta>0$ small enough. Hence, Φ is continuous on $C([0,1-\varepsilon])$. Let us assume for contradiction that $\Phi(g)\neq g$. Hence, there exists $\eta>0$ and $x_0\in[0,1-\varepsilon]$ such that $|\Phi(g)(x_0)-g(x_0)|>\eta$. There exists $\delta>0$ such that $\|\Phi(\varphi)-\Phi(g)\|<\frac{1}{3}\eta$ whenever $\|\varphi-g\|<\delta$. Take n_0 so large that $\|f_n-g\|<\min\{\delta,\frac{1}{3}\eta\}$ for all $n\geq n_0$. Hence, $\|f_{n+1}-\Phi(g)\|=\|\Phi(f_n)-\Phi(g)\|<\frac{1}{3}\eta$. On the other hand, we have $|f_{n+1}(x_0)-\Phi(g)(x_0)|>|\Phi(g)(x_0)-g(x_0)|-|g(x_0)-f_{n+1}(x_0)|>\eta-\frac{1}{3}\eta=\frac{2}{3}\eta$, contradiction. So, $\Phi(g)=g$.

Since f is the only fixed point of Φ in $\{\varphi\in C([0,1-\varepsilon]):\ \varphi(0)=1\}$, we have $g=f$ on $[0,1-\varepsilon]$. Since $\varepsilon\in(0,1)$ was arbitrary, we have $\lim_{n\to\infty}f_n(x)=\frac{1}{1-x}$ for all $x\in[0,1)$.